Additional exercises to practise induction

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The purpose of the exercises below is to practice induction. Although some of the exercises below can be solved (sometimes even more easily) without using induction, try nevertheless always to (also) give an inductive argument.

1. Prove by induction: $\forall_n[n \in \mathbb{N} : \sum_{i=1}^{n} 2(3^{i-1}) = 3^n - 1]$;
2. Prove by induction: $\forall_n[n \in \mathbb{N} \wedge n \geq 1 : \sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}]$.
3. Prove by induction: $\forall_n[n \in \mathbb{N} \wedge n > 4 : n^2 < 2^n]$.
4. Prove by induction: $\forall_n[\sum_{k=1}^{n} \frac{1}{k^2} \leq 2 - \frac{1}{n}]$.
5. Prove by induction that $\sum_{i=1}^{n} i^k \leq \frac{1}{2}n^k(n+1)$ for all $k \geq 1$ and $n \geq 0$.
   (You may use without proof that $\sum_{i=1}^{n} i = \frac{1}{2}n(n+1)$.)
6. Prove by induction on $n \in \mathbb{N}$ that $n^3 + 2n$ is divisible by 3.
7. Prove by induction on $n \in \mathbb{N}$ that $n^5 - n$ is divisible by 5.
8. Prove by induction that the sum of the cubes of three successive natural numbers is divisible by 9.
9. Prove that any integer postage greater than 34 cents can be formed using only 5-cent and 9-cent stamps.
10. Prove that any integer postage greater than 5 cents can be formed by using only 2-cent and 7-cent stamps.
11. Prove by induction that for all $n \in \mathbb{N}$ and all even $m \in \mathbb{N}$, an $n \times m$ chessboard has exactly the same number of black squares and white squares.
12. Prove by induction that for all odd $n, m \in \mathbb{N}$, an $n \times m$ chessboard has all four corner squares coloured the same.
13. Prove by induction that for all odd $n, m \in \mathbb{N}$, an $n \times m$ chessboard with the corner squares coloured white has one more white square than black squares.
14. Prove that a decimal number is divisible by 3 iff the sum of its digits is divisible by 3.

    (Suggestion: First argue that it is enough to establish the existence of a natural number k such that $d_0 d_1 \ldots d_n = 3 \cdot k + d_0 + d_1 + \cdots + d_n$; then prove this by induction.)
15. Let X be a finite set. Prove, by induction on $\#(X)$, that $\#(\mathcal{P}(X)) = 2^{\#(X)}$.