Exercise 14.9(a)

Show with a *derivation* that the formula

$$(P \Leftrightarrow Q) \Rightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$$

is a tautology.

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A *derivation* is a special kind of formal proof, constructed according to the rules listed in the Tables for Part II on pp. 375–380 of the book.

Exercise 14.9(a)

Show with a *derivation* that the formula

$$(P \Leftrightarrow Q) \Rightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$$

is a tautology.

A *derivation* is a special kind of formal proof, constructed according to the rules listed in the Tables for Part II on pp. 375–380 of the book.

The construction of a derivation starts with writing the goal (i.e., the formula that is to be proved) at the bottom. (Of course, in reality, we do not know in advance that the proof will take 17 lines.)

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(17) $(P \Leftrightarrow Q) \Rightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$

Exercise 14.9(a)

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The construction of a derivation starts with writing the goal (i.e., the formula that is to be proved) at the bottom. (Of course, in reality, we do not know in advance that the proof will take 17 lines.)

The main strategy is to try and simplify the goal until it becomes (logically) simple (in practice, a formula without connectives or quantifiers).

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(17) $(P \Leftrightarrow Q) \Rightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$

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Show with a *derivation* that the formula

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The main strategy is to try and simplify the goal until it becomes (logically) simple (in practice, a formula without connectives or quantifiers).

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(17) $(P \Leftrightarrow Q) \Rightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$

Exercise 14.9(a)

Show with a *derivation* that the formula

$$(P \Leftrightarrow Q) \Rightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$$

is a tautology.

Note that the derivation rules presented in Part II of the book may not be applied to subformulas. So the main symbol of the formula dictates which rule(s) can be applied. The main symbol in the present goal is $\Rightarrow$.

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(17) $(P \Leftrightarrow Q) \Rightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$

Exercise 14.9(a)

Show with a *derivation* that the formula

$$(P \Leftrightarrow Q) \Rightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$$

is a tautology.

Note that the derivation rules presented in Part II of the book may not be applied to subformulas. So the main symbol of the formula dictates which rule(s) can be applied. The main symbol in the present goal is $\Rightarrow$.

The only rule for $\Rightarrow$ (when it is the main symbol of a goal), is the following rule:

(17) $(P \Leftrightarrow Q) \Rightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$

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(17) $(P \Leftrightarrow Q) \Rightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$

Exercise 14.9(a)

Show with a *derivation* that the formula

$$(P \Leftrightarrow Q) \Rightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$$

is a tautology.

Note that the derivation rules presented in Part II of the book may not be applied to subformulas. So the main symbol of the formula dictates which rule(s) can be applied. The main symbol in the present goal is $\Rightarrow$.

The only rule for $\Rightarrow$ (when it is the main symbol of a goal), is the following rule:

$\Rightarrow$-introduction:

Assume:

(k) P

$\vdots$

$(\ell - 1)$ Q

{ $\Rightarrow$-intro on (k) and $(\ell - 1)$: }

(ℓ) $P \Rightarrow Q$

(17) $(P \Leftrightarrow Q) \Rightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$

Exercise 14.9(a)

Show with a *derivation* that the formula

$$(P \Leftrightarrow Q) \Rightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$$

is a tautology.

Note that the derivation rules presented in Part II of the book may not be applied to subformulas. So the main symbol of the formula dictates which rule(s) can be applied. The main symbol in the present goal is $\Rightarrow$.

The only rule for $\Rightarrow$ (when it is the main symbol of a goal), is the following rule:

$\Rightarrow$-introduction:

Assume:

(k) P

$\vdots$

$(\ell - 1)$ Q

$\{ \Rightarrow$-intro on (k) and $(\ell - 1)$: $\}$

(ℓ) $P \Rightarrow Q$

We can apply this rule to the goal by substituting $(P \Leftrightarrow Q)$ for P and $((P \wedge Q) \vee (\neg P \wedge \neg Q))$ for Q in the rule.

Exercise 14.9(a)

Show with a *derivation* that the formula

$$(P \Leftrightarrow Q) \Rightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$$

is a tautology.

{ Assume }

(1) $P \Leftrightarrow Q$

(2)

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(16) $(P \wedge Q) \vee (\neg P \wedge \neg Q)$

{ $\Rightarrow$-intro on (1) and (16) }

(17) $(P \Leftrightarrow Q) \Rightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$

Note that the derivation rules presented in Part II of the book may not be applied to subformulas. So the main symbol of the formula dictates which rule(s) can be applied. The main symbol in the present goal is $\Rightarrow$.

The only rule for $\Rightarrow$ (when it is the main symbol of a goal), is the following rule:

$\Rightarrow$-introduction:

Assume:

(k) P

$\vdots$

($\ell - 1$) Q

{ $\Rightarrow$-intro on (k) and ($\ell - 1$): }

(ℓ) $P \Rightarrow Q$

We can apply this rule to the goal by substituting $(P \Leftrightarrow Q)$ for P and $((P \wedge Q) \vee (\neg P \wedge \neg Q))$ for Q in the rule.

{ Assume }

(1) $P \Leftrightarrow Q$

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(16) $(P \wedge Q) \vee (\neg P \wedge \neg Q)$

{ $\Rightarrow$-intro on (1) and (16) }

(17) $(P \Leftrightarrow Q) \Rightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$

Exercise 14.9(a)

Show with a *derivation* that the formula

$$(P \Leftrightarrow Q) \Rightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$$

is a tautology.

{ Assume }

(1) $P \Leftrightarrow Q$

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(16) $(P \wedge Q) \vee (\neg P \wedge \neg Q)$

{ $\Rightarrow$-intro on (1) and (16) }

(17) $(P \Leftrightarrow Q) \Rightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$

Exercise 14.9(a)

Show with a *derivation* that the formula

$$(P \Leftrightarrow Q) \Rightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$$

is a tautology.

The original goal has now been simplified; the new goal is $(P \wedge Q) \vee (\neg P \wedge \neg Q)$.

{ Assume }

(1) $P \Leftrightarrow Q$

(2)

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(16) $(P \wedge Q) \vee (\neg P \wedge \neg Q)$

{ $\Rightarrow$-intro on (1) and (16) }

(17) $(P \Leftrightarrow Q) \Rightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$

Exercise 14.9(a)

Show with a *derivation* that the formula

$$(P \Leftrightarrow Q) \Rightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$$

is a tautology.

The original goal has now been simplified; the new goal is $(P \wedge Q) \vee (\neg P \wedge \neg Q)$.

The main symbol in the new goal is $\vee$; we have the following rules for formulas with $\vee$ as the main symbol of the goal:

{ Assume }

(1) $\boxed{P \Leftrightarrow Q}$

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(16) $(P \wedge Q) \vee (\neg P \wedge \neg Q)$

{ $\Rightarrow$-intro on (1) and (16) }

(17) $(P \Leftrightarrow Q) \Rightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$

Exercise 14.9(a)

Show with a *derivation* that the formula

$$(P \Leftrightarrow Q) \Rightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$$

is a tautology.

The original goal has now been simplified; the new goal is $(P \wedge Q) \vee (\neg P \wedge \neg Q)$.

The main symbol in the new goal is $\vee$; we have the following rules for formulas with $\vee$ as the main symbol of the goal:

$\vee$-introduction:

Assume:

(k) $\boxed{\neg P}$

$\vdots$

($\ell - 1$) Q

{ $\vee$-intro on (k) and ($\ell - 1$): }

(ℓ) $P \vee Q$

Assume:

(k) $\boxed{\neg Q}$

$\vdots$

($\ell - 1$) P

{ $\vee$-intro on (k) and ($\ell - 1$): }

(ℓ) $P \vee Q$

{ Assume }

(1) $\boxed{P \Leftrightarrow Q}$

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(16) $(P \wedge Q) \vee (\neg P \wedge \neg Q)$

{ $\Rightarrow$-intro on (1) and (16) }

(17) $(P \Leftrightarrow Q) \Rightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$

Exercise 14.9(a)

Show with a *derivation* that the formula

$$(P \Leftrightarrow Q) \Rightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$$

is a tautology.

The original goal has now been simplified; the new goal is $(P \wedge Q) \vee (\neg P \wedge \neg Q)$.

The main symbol in the new goal is $\vee$; we have the following rules for formulas with $\vee$ as the main symbol of the goal:

$\vee$-introduction:

Assume:

(k) $\boxed{\neg P}$

⋮

$(\ell - 1)$ Q

{ $\vee$-intro on (k) and $(\ell - 1)$: }

(ℓ) $P \vee Q$

Assume:

(k) $\boxed{\neg Q}$

⋮

$(\ell - 1)$ P

{ $\vee$-intro on (k) and $(\ell - 1)$: }

(ℓ) $P \vee Q$

We choose the left-hand side rule, substituting $P \wedge Q$ for P and $\neg P \wedge \neg Q$ for Q.

{ Assume }

(1) $P \Leftrightarrow Q$

{ Assume }

(2) $\neg(P \wedge Q)$

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(15) $\neg P \wedge \neg Q$

{ $\vee$-intro on (2) and (15) }

(16) $(P \wedge Q) \vee (\neg P \wedge \neg Q)$

{ $\Rightarrow$-intro on (1) and (16) }

(17) $(P \Leftrightarrow Q) \Rightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$

Exercise 14.9(a)

Show with a *derivation* that the formula

$$(P \Leftrightarrow Q) \Rightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$$

is a tautology.

The original goal has now been simplified; the new goal is $(P \wedge Q) \vee (\neg P \wedge \neg Q)$.

The main symbol in the new goal is $\vee$; we have the following rules for formulas with $\vee$ as the main symbol of the goal:

$\vee$-introduction:

Assume:

(k) $\neg P$

$\vdots$

($\ell - 1$) Q

{ $\vee$-intro on (k) and ($\ell - 1$): }

(ℓ) $P \vee Q$

Assume:

(k) $\neg Q$

$\vdots$

($\ell - 1$) P

{ $\vee$-intro on (k) and ($\ell - 1$): }

(ℓ) $P \vee Q$

We choose the left-hand side rule, substituting $P \wedge Q$ for P and $\neg P \wedge \neg Q$ for Q.

Exercise 14.9(a)

Show with a *derivation* that the formula

$$(P \Leftrightarrow Q) \Rightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$$

is a tautology.

The new goal is $\neg P \wedge \neg Q$, with $\wedge$ as main symbol.

{ Assume }

(1) $P \Leftrightarrow Q$

{ Assume }

(2) $\neg(P \wedge Q)$

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(15) $\neg P \wedge \neg Q$

{ $\vee$-intro on (2) and (15) }

(16) $(P \wedge Q) \vee (\neg P \wedge \neg Q)$

{ $\Rightarrow$-intro on (1) and (16) }

(17) $(P \Leftrightarrow Q) \Rightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$

Exercise 14.9(a)

Show with a *derivation* that the formula

$$(P \Leftrightarrow Q) \Rightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$$

is a tautology.

The new goal is $\neg P \wedge \neg Q$, with $\wedge$ as main symbol.

We have the following rule for simplifying a goal with $\wedge$ as its main symbol:

{ Assume }

(1) $P \Leftrightarrow Q$

{ Assume }

(2) $\neg(P \wedge Q)$

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(15) $\neg P \wedge \neg Q$

{ $\vee$-intro on (2) and (15) }

(16) $(P \wedge Q) \vee (\neg P \wedge \neg Q)$

{ $\Rightarrow$-intro on (1) and (16) }

(17) $(P \Leftrightarrow Q) \Rightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$

{ Assume }

(1) $P \Leftrightarrow Q$

{ Assume }

(2) $\neg(P \wedge Q)$

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(15) $\neg P \wedge \neg Q$

{ $\vee$-intro on (2) and (15) }

(16) $(P \wedge Q) \vee (\neg P \wedge \neg Q)$

{ $\Rightarrow$-intro on (1) and (16) }

(17) $(P \Leftrightarrow Q) \Rightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$

Exercise 14.9(a)

Show with a *derivation* that the formula

$$(P \Leftrightarrow Q) \Rightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$$

is a tautology.

The new goal is $\neg P \wedge \neg Q$, with $\wedge$ as main symbol.

We have the following rule for simplifying a goal with $\wedge$ as its main symbol:

$\wedge$-introduction:

$\vdots$

(k) P

$\vdots$

(ℓ) Q

{ $\wedge$-intro on (k) and (ℓ): }

(m) $P \wedge Q$

{ Assume }

(1) $P \Leftrightarrow Q$

{ Assume }

(2) $\neg(P \wedge Q)$

(3)

(4)

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(15) $\neg P \wedge \neg Q$

{ $\vee$-intro on (2) and (15) }

(16) $(P \wedge Q) \vee (\neg P \wedge \neg Q)$

{ $\Rightarrow$-intro on (1) and (16) }

(17) $(P \Leftrightarrow Q) \Rightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$

Exercise 14.9(a)

Show with a *derivation* that the formula

$$(P \Leftrightarrow Q) \Rightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$$

is a tautology.

The new goal is $\neg P \wedge \neg Q$, with $\wedge$ as main symbol.

We have the following rule for simplifying a goal with $\wedge$ as its main symbol:

$\wedge$-introduction:

$\vdots$

(k) P

$\vdots$

(ℓ) Q

{ $\wedge$-intro on (k) and (ℓ): }

(m) $P \wedge Q$

We can apply this rule (substituting $\neg P$ for P and $\neg Q$ for Q), and the application generates two new subgoals ($\neg P$ and $\neg Q$), which have to be derived separately.

{ Assume }

(1) $P \Leftrightarrow Q$

{ Assume }

(2) $\neg(P \wedge Q)$

(3)

(4)

(5)

(6)

(7)

(8) $\neg P$

(9)

(10)

(11)

(12)

(13)

(14) $\neg Q$

{ $\wedge$-intro on (8) and (14) }

(15) $\neg P \wedge \neg Q$

{ $\vee$-intro on (2) and (15) }

(16) $(P \wedge Q) \vee (\neg P \wedge \neg Q)$

{ $\Rightarrow$-intro on (1) and (16) }

(17) $(P \Leftrightarrow Q) \Rightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$

Exercise 14.9(a)

Show with a *derivation* that the formula

$$(P \Leftrightarrow Q) \Rightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$$

is a tautology.

The new goal is $\neg P \wedge \neg Q$, with $\wedge$ as main symbol.

We have the following rule for simplifying a goal with $\wedge$ as its main symbol:

$\wedge$-introduction:

$\vdots$

(k) P

$\vdots$

(ℓ) Q

{ $\wedge$-intro on (k) and (ℓ): }

(m) $P \wedge Q$

We can apply this rule (substituting $\neg P$ for P and $\neg Q$ for Q), and the application generates two new subgoals ($\neg P$ and $\neg Q$), which have to be derived separately.

Exercise 14.9(a)

Show with a *derivation* that the formula

$$(P \Leftrightarrow Q) \Rightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$$

is a tautology.

Let's first take care of the first subgoal $\neg P$, which has $\neg$ as its main symbol.

{ Assume }

(1) $P \Leftrightarrow Q$

{ Assume }

(2) $\neg(P \wedge Q)$

(3)

(4)

(5)

(6)

(7)

(8) $\neg P$

(9)

(10)

(11)

(12)

(13)

(14) $\neg Q$

{ $\wedge$-intro on (8) and (14) }

(15) $\neg P \wedge \neg Q$

{ $\vee$-intro on (2) and (15) }

(16) $(P \wedge Q) \vee (\neg P \wedge \neg Q)$

{ $\Rightarrow$-intro on (1) and (16) }

(17) $(P \Leftrightarrow Q) \Rightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$

{ Assume }

(1) $P \Leftrightarrow Q$

{ Assume }

(2) $\neg(P \wedge Q)$

(3)

(4)

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(8) $\neg P$

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(14) $\neg Q$

{ $\wedge$-intro on (8) and (14) }

(15) $\neg P \wedge \neg Q$

{ $\vee$-intro on (2) and (15) }

(16) $(P \wedge Q) \vee (\neg P \wedge \neg Q)$

{ $\Rightarrow$-intro on (1) and (16) }

(17) $(P \Leftrightarrow Q) \Rightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$

Exercise 14.9(a)

Show with a *derivation* that the formula

$$(P \Leftrightarrow Q) \Rightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$$

is a tautology.

Let's first take care of the first subgoal $\neg P$, which has $\neg$ as its main symbol.

We have the following rule for simplifying a goal with $\neg$ as its main symbol:

{ Assume }

(1) $P \Leftrightarrow Q$

{ Assume }

(2) $\neg(P \wedge Q)$

(3)

(4)

(5)

(6)

(7)

(8) $\neg P$

(9)

(10)

(11)

(12)

(13)

(14) $\neg Q$

{ $\wedge$-intro on (8) and (14) }

(15) $\neg P \wedge \neg Q$

{ $\vee$-intro on (2) and (15) }

(16) $(P \wedge Q) \vee (\neg P \wedge \neg Q)$

{ $\Rightarrow$-intro on (1) and (16) }

(17) $(P \Leftrightarrow Q) \Rightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$

Exercise 14.9(a)

Show with a *derivation* that the formula

$$(P \Leftrightarrow Q) \Rightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$$

is a tautology.

Let's first take care of the first subgoal $\neg P$, which has $\neg$ as its main symbol.

We have the following rule for simplifying a goal with $\neg$ as its main symbol:

$\neg$-introduction:

Assume:

(k) P

⋮

($\ell - 1$) `False`

{ $\neg$-intro on (k) and ($\ell - 1$): }

(ℓ) $\neg P$

{ Assume }

(1) $P \Leftrightarrow Q$

{ Assume }

(2) $\neg(P \wedge Q)$

(3)

(4)

(5)

(6)

(7)

(8) $\neg P$

(9)

(10)

(11)

(12)

(13)

(14) $\neg Q$

{ $\wedge$-intro on (8) and (14) }

(15) $\neg P \wedge \neg Q$

{ $\vee$-intro on (2) and (15) }

(16) $(P \wedge Q) \vee (\neg P \wedge \neg Q)$

{ $\Rightarrow$-intro on (1) and (16) }

(17) $(P \Leftrightarrow Q) \Rightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$

Exercise 14.9(a)

Show with a *derivation* that the formula

$$(P \Leftrightarrow Q) \Rightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$$

is a tautology.

Let's first take care of the first subgoal $\neg P$, which has $\neg$ as its main symbol.

We have the following rule for simplifying a goal with $\neg$ as its main symbol:

$\neg$-introduction:

Assume:

(k) P

⋮

$(\ell - 1)$ False

{ $\neg$-intro on (k) and $(\ell - 1)$: }

(ℓ) $\neg P$

According to the rule, a negation $\neg P$ is proved by assuming the negated formula P and deriving a contradiction (False).

Exercise 14.9(a)

Show with a *derivation* that the formula

$$(P \Leftrightarrow Q) \Rightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$$

is a tautology.

Let's first take care of the first subgoal $\neg P$, which has $\neg$ as its main symbol.

We have the following rule for simplifying a goal with $\neg$ as its main symbol:

¬-introduction:

Assume:

(k) P

⋮

$(\ell - 1)$ `False`

{ $\neg$-intro on (k) and $(\ell - 1)$: }

(ℓ) $\neg P$

According to the rule, a negation $\neg P$ is proved by assuming the negated formula P and deriving a contradiction (`False`).

{ Assume }

(1) $P \Leftrightarrow Q$

{ Assume }

(2) $\neg(P \wedge Q)$

{ Assume }

(3) P

(4)

(5)

(6)

(7) `False`

{ $\neg$-intro on (3) and (7) }

(8) $\neg P$

(9)

(10)

(11)

(12)

(13)

(14) $\neg Q$

{ $\wedge$-intro on (8) and (14) }

(15) $\neg P \wedge \neg Q$

{ $\vee$-intro on (2) and (15) }

(16) $(P \wedge Q) \vee (\neg P \wedge \neg Q)$

{ $\Rightarrow$-intro on (1) and (16) }

(17) $(P \Leftrightarrow Q) \Rightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$

{ Assume }

(1) $P \Leftrightarrow Q$

{ Assume }

(2) $\neg(P \wedge Q)$

{ Assume }

(3) P

(4)

(5)

(6)

(7) False

{ $\neg$-intro on (3) and (7) }

(8) $\neg P$

(9)

(10)

(11)

(12)

(13)

(14) $\neg Q$

{ $\wedge$-intro on (8) and (14) }

(15) $\neg P \wedge \neg Q$

{ $\vee$-intro on (2) and (15) }

(16) $(P \wedge Q) \vee (\neg P \wedge \neg Q)$

{ $\Rightarrow$-intro on (1) and (16) }

(17) $(P \Leftrightarrow Q) \Rightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$

Exercise 14.9(a)

Show with a *derivation* that the formula

$$(P \Leftrightarrow Q) \Rightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$$

is a tautology.

The new subgoal `False` cannot be simplified further. (Although there is a rule called `False`-intro, we recommend *against* applying it; the difficulty of this rule is that it is hard to 'predict' what is a suitable formula to substitute for P in this rule.) Therefore, we will now do some *forward reasoning* and consider whether we can derive `False` from the available assumptions ($P \Leftrightarrow Q$, $\neg(P \wedge Q)$ and P).

Exercise 14.9(a)

Show with a *derivation* that the formula

$$(P \Leftrightarrow Q) \Rightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$$

is a tautology.

The new subgoal `False` cannot be simplified further. (Although there is a rule called `False`-intro, we recommend *against* applying it; the difficulty of this rule is that it is hard to 'predict' what is a suitable formula to substitute for P in this rule.) Therefore, we will now do some *forward reasoning* and consider whether we can derive `False` from the available assumptions ($P \Leftrightarrow Q$, $\neg(P \wedge Q)$ and P).

Note that we can infer $P \Rightarrow Q$ from $P \Leftrightarrow Q$, using one of the rules for $\Leftrightarrow$-elimination, and then do $\Rightarrow$-elimination using $P \Rightarrow Q$ and P to conclude Q. Then we have both P and Q so, by $\wedge$-introduction we obtain $P \wedge Q$, and then, by $\neg$-elimination we infer a contradiction `False` from $\neg(P \wedge Q)$ and $P \wedge Q$.

{ Assume }
(1) $P \Leftrightarrow Q$
{ Assume }
(2) $\neg(P \wedge Q)$
{ Assume }
(3) P
(4)
(5)
(6)
(7) `False`
{ $\neg$-intro on (3) and (7) }
(8) $\neg P$
(9)
(10)
(11)
(12)
(13)
(14) $\neg Q$
{ $\wedge$-intro on (8) and (14) }
(15) $\neg P \wedge \neg Q$
{ $\vee$-intro on (2) and (15) }
(16) $(P \wedge Q) \vee (\neg P \wedge \neg Q)$
{ $\Rightarrow$-intro on (1) and (16) }
(17) $(P \Leftrightarrow Q) \Rightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$

{ Assume }
(1) $P \Leftrightarrow Q$
{ Assume }
(2) $\neg(P \wedge Q)$
{ Assume }
(3) P
(4)
(5)
(6)
(7) `False`
{ $\neg$-intro on (3) and (7) }
(8) $\neg P$
(9)
(10)
(11)
(12)
(13)
(14) $\neg Q$
{ $\wedge$-intro on (8) and (14) }
(15) $\neg P \wedge \neg Q$
{ $\vee$-intro on (2) and (15) }
(16) $(P \wedge Q) \vee (\neg P \wedge \neg Q)$
{ $\Rightarrow$-intro on (1) and (16) }
(17) $(P \Leftrightarrow Q) \Rightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$

Exercise 14.9(a)

Show with a *derivation* that the formula

$$(P \Leftrightarrow Q) \Rightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$$

is a tautology.

The new subgoal `False` cannot be simplified further. (Although there is a rule called `False`-intro, we recommend *against* applying it; the difficulty of this rule is that it is hard to 'predict' what is a suitable formula to substitute for P in this rule.) Therefore, we will now do some *forward reasoning* and consider whether we can derive `False` from the available assumptions ($P \Leftrightarrow Q$, $\neg(P \wedge Q)$ and P).

Note that we can infer $P \Rightarrow Q$ from $P \Leftrightarrow Q$, using one of the rules for $\Leftrightarrow$-elimination, and then do $\Rightarrow$-elimination using $P \Rightarrow Q$ and P to conclude Q. Then we have both P and Q so, by $\wedge$-introduction we obtain $P \wedge Q$, and then, by $\neg$-elimination we infer a contradiction `False` from $\neg(P \wedge Q)$ and $P \wedge Q$.

This is worked out in detail on the following slides.

Exercise 14.9(a)

Show with a *derivation* that the formula

$$(P \Leftrightarrow Q) \Rightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$$

is a tautology.

{ Assume }

(1) $P \Leftrightarrow Q$

{ Assume }

(2) $\neg(P \wedge Q)$

{ Assume }

(3) P

(4)

(5)

(6)

(7) False

{ $\neg$-intro on (3) and (7) }

(8) $\neg P$

(9)

(10)

(11)

(12)

(13)

(14) $\neg Q$

{ $\wedge$-intro on (8) and (14) }

(15) $\neg P \wedge \neg Q$

{ $\vee$-intro on (2) and (15) }

(16) $(P \wedge Q) \vee (\neg P \wedge \neg Q)$

{ $\Rightarrow$-intro on (1) and (16) }

(17) $(P \Leftrightarrow Q) \Rightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$

First we infer $P \Rightarrow Q$ from $P \Leftrightarrow Q$, using the left-hand side rule below:

$\Leftrightarrow$-elimination:

	$\vert\vert\vert$
(k)	$P \Leftrightarrow Q$
	$\vert\vert\vert$
	{ $\Leftrightarrow$-elim on (k): }
(ℓ)	$P \Rightarrow Q$

	$\vert\vert\vert$
(k)	$P \Leftrightarrow Q$
	$\vert\vert\vert$
	{ $\Leftrightarrow$-elim on (k): }
(ℓ)	$Q \Rightarrow P$

{ Assume }

(1) $\boxed{P \Leftrightarrow Q}$

{ Assume }

(2) $\boxed{\neg(P \wedge Q)}$

{ Assume }

(3) $\boxed{P}$

{ $\Leftrightarrow$-elim on (1) }

(4) $P \Rightarrow Q$

(5)

(6)

(7) `False`

{ $\neg$-intro on (3) and (7) }

(8) $\neg P$

(9)

(10)

(11)

(12)

(13)

(14) $\neg Q$

{ $\wedge$-intro on (8) and (14) }

(15) $\neg P \wedge \neg Q$

{ $\vee$-intro on (2) and (15) }

(16) $(P \wedge Q) \vee (\neg P \wedge \neg Q)$

{ $\Rightarrow$-intro on (1) and (16) }

(17) $(P \Leftrightarrow Q) \Rightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$

Exercise 14.9(a)

Show with a *derivation* that the formula

$$(P \Leftrightarrow Q) \Rightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$$

is a tautology.

First we infer $P \Rightarrow Q$ from $P \Leftrightarrow Q$, using the left-hand side rule below:

$\Leftrightarrow$-elimination:

	$\|\|\|$
(k)	$P \Leftrightarrow Q$
	$\|\|\|$
	{ $\Leftrightarrow$-elim on (k): }
(ℓ)	$P \Rightarrow Q$

	$\|\|\|$
(k)	$P \Leftrightarrow Q$
	$\|\|\|$
	{ $\Leftrightarrow$-elim on (k): }
(ℓ)	$Q \Rightarrow P$

Exercise 14.9(a)

Show with a *derivation* that the formula

$$(P \Leftrightarrow Q) \Rightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$$

is a tautology.

	{ Assume }
(1)	$P \Leftrightarrow Q$
	{ Assume }
(2)	$\neg(P \wedge Q)$
	{ Assume }
(3)	P
	{ $\Leftrightarrow$-elim on (1) }
(4)	$P \Rightarrow Q$
(5)	
(6)	
(7)	False
	{ $\neg$-intro on (3) and (7) }
(8)	$\neg P$
(9)	
(10)	
(11)	
(12)	
(13)	
(14)	$\neg Q$
	{ $\wedge$-intro on (8) and (14) }
(15)	$\neg P \wedge \neg Q$
	{ $\vee$-intro on (2) and (15) }
(16)	$(P \wedge Q) \vee (\neg P \wedge \neg Q)$
	{ $\Rightarrow$-intro on (1) and (16) }
(17)	$(P \Leftrightarrow Q) \Rightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$

Then, we infer Q from $P \Rightarrow Q$ and P using the following rule:

$\Rightarrow$-elimination:

	$\vdots$
(k)	$P \Rightarrow Q$
	$\vdots$
(ℓ)	P
	$\vdots$
	{ $\Rightarrow$-elim on (k) and (ℓ): }
(m)	Q

Exercise 14.9(a)

Show with a *derivation* that the formula

$$(P \Leftrightarrow Q) \Rightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$$

is a tautology.

{ Assume }
(1) $P \Leftrightarrow Q$
  { Assume }
(2) $\neg(P \wedge Q)$
    { Assume }
(3) P
      { $\Leftrightarrow$-elim on (1) }
(4) $P \Rightarrow Q$
      { $\Rightarrow$-elim on (4) and (3) }
(5) Q
(6)
(7) False
    { $\neg$-intro on (3) and (7) }
(8) $\neg P$
(9)
(10)
(11)
(12)
(13)
(14) $\neg Q$
    { $\wedge$-intro on (8) and (14) }
(15) $\neg P \wedge \neg Q$
  { $\vee$-intro on (2) and (15) }
(16) $(P \wedge Q) \vee (\neg P \wedge \neg Q)$
{ $\Rightarrow$-intro on (1) and (16) }
(17) $(P \Leftrightarrow Q) \Rightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$

Then, we infer Q from $P \Rightarrow Q$ and P using the following rule:

$\Rightarrow$-elimination:

	$\vert\vert\vert$
(k)	$P \Rightarrow Q$
	$\vert\vert\vert$
(ℓ)	P
	$\vert\vert\vert$
	{ $\Rightarrow$-elim on (k) and (ℓ): }
(m)	Q

{ Assume }

(1) $P \Leftrightarrow Q$

{ Assume }

(2) $\neg(P \wedge Q)$

{ Assume }

(3) P

{ $\Leftrightarrow$-elim on (1) }

(4) $P \Rightarrow Q$

{ $\Rightarrow$-elim on (4) and (3) }

(5) Q

(6)

(7) False

{ $\neg$-intro on (3) and (7) }

(8) $\neg P$

(9)

(10)

(11)

(12)

(13)

(14) $\neg Q$

{ $\wedge$-intro on (8) and (14) }

(15) $\neg P \wedge \neg Q$

{ $\vee$-intro on (2) and (15) }

(16) $(P \wedge Q) \vee (\neg P \wedge \neg Q)$

{ $\Rightarrow$-intro on (1) and (16) }

(17) $(P \Leftrightarrow Q) \Rightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$

Exercise 14.9(a)

Show with a *derivation* that the formula

$$(P \Leftrightarrow Q) \Rightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$$

is a tautology.

Then, we infer $P \wedge Q$ from P and Q using

$\wedge$-introduction:

$\vdots$

(k) P

$\vdots$

(ℓ) Q

{ $\wedge$-intro on (k) and (ℓ): }

(m) $P \wedge Q$

(The rule for $\wedge$-introduction has been presented as a rule that is used in backward reasoning, but here it is used in our forward reasoning. Alternatively, we may view the step as the introduction of a new goal $P \wedge Q$, with the purpose of doing a $\neg$-elimination later, and the goal can immediately be inferred, since P and Q are already available.)

{ Assume }

(1) $P \Leftrightarrow Q$

{ Assume }

(2) $\neg(P \wedge Q)$

{ Assume }

(3) P

{ $\Leftrightarrow$-elim on (1) }

(4) $P \Rightarrow Q$

{ $\Rightarrow$-elim on (4) and (3) }

(5) Q

{ $\wedge$-intro on (3) and (5) }

(6) $P \wedge Q$

(7) `False`

{ $\neg$-intro on (3) and (7) }

(8) $\neg P$

(9)

(10)

(11)

(12)

(13)

(14) $\neg Q$

{ $\wedge$-intro on (8) and (14) }

(15) $\neg P \wedge \neg Q$

{ $\vee$-intro on (2) and (15) }

(16) $(P \wedge Q) \vee (\neg P \wedge \neg Q)$

{ $\Rightarrow$-intro on (1) and (16) }

(17) $(P \Leftrightarrow Q) \Rightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$

Exercise 14.9(a)

Show with a *derivation* that the formula

$$(P \Leftrightarrow Q) \Rightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$$

is a tautology.

Then, we infer $P \wedge Q$ from P and Q using

$\wedge$-introduction:

$\vdots$

(k) P

$\vdots$

(ℓ) Q

{ $\wedge$-intro on (k) and (ℓ): }

(m) $P \wedge Q$

(The rule for $\wedge$-introduction has been presented as a rule that is used in backward reasoning, but here it is used in our forward reasoning. Alternatively, we may view the step as the introduction of a new goal $P \wedge Q$, with the purpose of doing a $\neg$-elimination later, and the goal can immediately be inferred, since P and Q are already available.)

{ Assume }

(1) $P \Leftrightarrow Q$

{ Assume }

(2) $\neg(P \wedge Q)$

{ Assume }

(3) P

{ $\Leftrightarrow$-elim on (1) }

(4) $P \Rightarrow Q$

{ $\Rightarrow$-elim on (4) and (3) }

(5) Q

{ $\wedge$-intro on (3) and (5) }

(6) $P \wedge Q$

(7) False

{ $\neg$-intro on (3) and (7) }

(8) $\neg P$

(9)

(10)

(11)

(12)

(13)

(14) $\neg Q$

{ $\wedge$-intro on (8) and (14) }

(15) $\neg P \wedge \neg Q$

{ $\vee$-intro on (2) and (15) }

(16) $(P \wedge Q) \vee (\neg P \wedge \neg Q)$

{ $\Rightarrow$-intro on (1) and (16) }

(17) $(P \Leftrightarrow Q) \Rightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$

Exercise 14.9(a)

Show with a *derivation* that the formula

$$(P \Leftrightarrow Q) \Rightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$$

is a tautology.

Then, we infer the required contradiction (False) from $\neg(P \wedge Q)$ and $P \wedge Q$ using

$\neg$-elimination:

$\|\|$

(k) $\neg P$

$\|\|$

(ℓ) P

$\|\|$

$\neg$-elim on (k) and (ℓ):

(m) False

Exercise 14.9(a)

Show with a *derivation* that the formula

$$(P \Leftrightarrow Q) \Rightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$$

is a tautology.

Then, we infer the required contradiction ($\texttt{False}$) from $\neg(P \wedge Q)$ and $P \wedge Q$ using

$\neg$-elimination:

	$\Vert\Vert$
(k)	$\neg P$
	$\Vert\Vert$
(ℓ)	P
	$\Vert\Vert$
	$\neg$-elim on (k) and (ℓ):
(m)	$\texttt{False}$

{ Assume }

(1) $P \Leftrightarrow Q$

{ Assume }

(2) $\neg(P \wedge Q)$

{ Assume }

(3) P

{ $\Leftrightarrow$-elim on (1) }

(4) $P \Rightarrow Q$

{ $\Rightarrow$-elim on (4) and (3) }

(5) Q

{ $\wedge$-intro on (3) and (5) }

(6) $P \wedge Q$

{ $\neg$-elim on (2) and (6) }

(7) $\texttt{False}$

{ $\neg$-intro on (3) and (7) }

(8) $\neg P$

(9)

(10)

(11)

(12)

(13)

(14) $\neg Q$

{ $\wedge$-intro on (8) and (14) }

(15) $\neg P \wedge \neg Q$

{ $\vee$-intro on (2) and (15) }

(16) $(P \wedge Q) \vee (\neg P \wedge \neg Q)$

{ $\Rightarrow$-intro on (1) and (16) }

(17) $(P \Leftrightarrow Q) \Rightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$

Exercise 14.9(a)

Show with a *derivation* that the formula

$$(P \Leftrightarrow Q) \Rightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$$

is a tautology.

We should now still take care of the other subgoal $\neg Q$. The derivation of $\neg Q$ from $P \Leftrightarrow Q$ and $\neg(P \wedge Q)$ is analogous to the derivation of $\neg P$ from $P \Leftrightarrow Q$ and $\neg(P \wedge Q)$. (Of course, it uses the right-hand side rule for $\Leftrightarrow$-elimination, and interchanges the roles of P and Q in the subderivation.)

We shall not discuss the separate steps in detail, and just add the missing steps to the derivation on the left.

NB: the formulas on lines (3)–(7) cannot be used in the derivation of $\neg Q$, for with the application of the $\neg$-intro rule used to conclude $\neg P$ on line (8), the assumption P (and all its consequences) are withdrawn.

{ Assume }
(1) $P \Leftrightarrow Q$
{ Assume }
(2) $\neg(P \wedge Q)$
{ Assume }
(3) P
{ $\Leftrightarrow$-elim on (1) }
(4) $P \Rightarrow Q$
{ $\Rightarrow$-elim on (4) and (3) }
(5) Q
{ $\wedge$-intro on (3) and (5) }
(6) $P \wedge Q$
{ $\neg$-elim on (2) and (6) }
(7) `False`
{ $\neg$-intro on (3) and (7) }
(8) $\neg P$
(9)
(10)
(11)
(12)
(13)
(14) $\neg Q$
{ $\wedge$-intro on (8) and (14) }
(15) $\neg P \wedge \neg Q$
{ $\vee$-intro on (2) and (15) }
(16) $(P \wedge Q) \vee (\neg P \wedge \neg Q)$
{ $\Rightarrow$-intro on (1) and (16) }
(17) $(P \Leftrightarrow Q) \Rightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$

Exercise 14.9(a)

Show with a *derivation* that the formula

$$(P \Leftrightarrow Q) \Rightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$$

is a tautology.

We should now still take care of the other subgoal $\neg Q$. The derivation of $\neg Q$ from $P \Leftrightarrow Q$ and $\neg(P \wedge Q)$ is analogous to the derivation of $\neg P$ from $P \Leftrightarrow Q$ and $\neg(P \wedge Q)$. (Of course, it uses the right-hand side rule for $\Leftrightarrow$-elimination, and interchanges the roles of P and Q in the subderivation.)

We shall not discuss the separate steps in detail, and just add the missing steps to the derivation on the left.

NB: the formulas on lines (3)–(7) cannot be used in the derivation of $\neg Q$, for with the application of the $\neg$-intro rule used to conclude $\neg P$ on line (8), the assumption P (and all its consequences) are withdrawn.

{ Assume }
(1) $P \Leftrightarrow Q$
{ Assume }
(2) $\neg(P \wedge Q)$
{ Assume }
(3) P
{ $\Leftrightarrow$-elim on (1) }
(4) $P \Rightarrow Q$
{ $\Rightarrow$-elim on (4) and (3) }
(5) Q
{ $\wedge$-intro on (3) and (5) }
(6) $P \wedge Q$
{ $\neg$-elim on (2) and (6) }
(7) False
{ $\neg$-intro on (3) and (7) }
(8) $\neg P$
{ Assume }
(9) Q
{ $\Leftrightarrow$-elim on (1) }
(10) $Q \Rightarrow P$
{ $\Rightarrow$-elim on (10) and (9) }
(11) P
{ $\wedge$-intro on (11) and (9) }
(12) $P \wedge Q$
{ $\neg$-elim on (2) and (12) }
(13) False
{ $\neg$-intro on (9) and (13) }
(14) $\neg Q$
{ $\wedge$-intro on (8) and (14) }
(15) $\neg P \wedge \neg Q$
{ $\vee$-intro on (2) and (15) }
(16) $(P \wedge Q) \vee (\neg P \wedge \neg Q)$
{ $\Rightarrow$-intro on (1) and (16) }
(17) $(P \Leftrightarrow Q) \Rightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$

Exercise 14.9(a)

Show with a *derivation* that the formula

$$(P \Leftrightarrow Q) \Rightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$$

is a tautology.

We have filled all 'gaps' in the derivation on the left (all subgoals have been derived), and there are no open assumptions (all flagpoles end with appropriate applications of introduction rules). Thus, we have obtained a derivation that shows that the formula

$$(P \Leftrightarrow Q) \Rightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$$

is a tautology.

{ Assume }
(1) $P \Leftrightarrow Q$
{ Assume }
(2) $\neg(P \wedge Q)$
{ Assume }
(3) P
{ $\Leftrightarrow$-elim on (1) }
(4) $P \Rightarrow Q$
{ $\Rightarrow$-elim on (4) and (3) }
(5) Q
{ $\wedge$-intro on (3) and (5) }
(6) $P \wedge Q$
{ $\neg$-elim on (2) and (6) }
(7) `False`
{ $\neg$-intro on (3) and (7) }
(8) $\neg P$
{ Assume }
(9) Q
{ $\Leftrightarrow$-elim on (1) }
(10) $Q \Rightarrow P$
{ $\Rightarrow$-elim on (10) and (9) }
(11) P
{ $\wedge$-intro on (11) and (9) }
(12) $P \wedge Q$
{ $\neg$-elim on (2) and (12) }
(13) `False`
{ $\neg$-intro on (9) and (13) }
(14) $\neg Q$
{ $\wedge$-intro on (8) and (14) }
(15) $\neg P \wedge \neg Q$
{ $\vee$-intro on (2) and (15) }
(16) $(P \wedge Q) \vee (\neg P \wedge \neg Q)$
{ $\Rightarrow$-intro on (1) and (16) }
(17) $(P \Leftrightarrow Q) \Rightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$