Example

Let $F : A \to B$ be a mapping, and let $S, T \subseteq A$; then

$$F(S) \backslash F(T) \subseteq F(S \backslash T) \ .$$

We first give a *derivation-style* proof of this property; then we shall also give a proof in natural language.

Example

Let $F : A \to B$ be a mapping, and let $S, T \subseteq A$; then

$$F(S) \setminus F(T) \subseteq F(S \setminus T) \ .$$

The goal is to establish that the set $F(S) \setminus F(T)$ is a subset of the set $F(S \setminus T)$.

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The goal is to establish that the set $F(S) \setminus F(T)$ is a subset of the set $F(S \setminus T)$.

According to the definition of the predicate $\subseteq$:

$$\boxed{A \subseteq B \stackrel{\text{def}}{=} \forall_x [x \in A : x \in B]}$$

we then need to establish that every element x in A is also in B.

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Note that, for the application of this general property in this particular proof, A has to be instantiated with $F(S) \backslash F(T)$, and B has to be instantiated with $F(S \backslash T)$.

We adopt the convention that elements of the domain of F are ranged over by $x, x_1, x_2, \ldots$, and the elements of the range of F are ranged over by $y, y_1, y_2, \ldots$. Since $F(S) \backslash F(T)$ and $F(S \backslash T)$ are subsets of the range of F, we rename the bound variable x to y in the definition of $\subseteq$ above before we apply it.

var y; $y \in F(S) \setminus F(T)$

$y \in F(S \setminus T)$

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Let $F : A \rightarrow B$ be a mapping, and let $S, T \subseteq A$; then

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Let $F : A \rightarrow B$ be a mapping, and let $S, T \subseteq A$; then

$$F(S) \setminus F(T) \subseteq F(S \setminus T) \ .$$

The new goal is now to prove that $y \in F(S \setminus T)$, i.e., that y is in the F-image of the set $S \setminus T$.

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There are two properties for reasoning about the image of a mapping:

Properties of 'image':

$x \in A' \overset{val}{\models} F(x) \in F(A')$

$y \in F(A') \overset{val}{==} \exists_x [x \in A' : F(x) = y]$

$\mathbf{var}\ y;\ y \in F(S) \setminus F(T)$

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Only the second property applies; we should then instantiate A' with $S \setminus T$.

var y; $y \in F(S) \setminus F(T)$

$\exists_x[x \in S \setminus T : F(x) = y]$

{ Property of image:}

$y \in F(S \setminus T)$

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Only the second property applies; we should then instantiate A' with $S \setminus T$.

The general strategy for proving an existential quantification is to *wait for a witness*.

var y; $y \in F(S) \setminus F(T)$

$\exists_x[x \in S \setminus T : F(x) = y]$

{ Property of image:}

$y \in F(S\setminus T)$

$F(S)\setminus F(T) \subseteq F(S\setminus T)$

Example

Let $F : A \to B$ be a mapping, and let $S, T \subseteq A$; then

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So we continue to consider the assumption $y \in F(X)\setminus F(T)$, to find out if it yields an appropriate witness.

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$\mathbf{var}\ y;\ y \in F(S) \setminus F(T)$

{ Property of $\setminus$: }

$y \in F(S) \wedge \neg(y \in F(T))$

{ Property of image: }

$\exists_x[x \in S : F(x) = y]$

$\exists_x[x \in S \setminus T : F(x) = y]$

{ Property of image:}

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The resulting existential quantification yields a potential witness via $\exists^*$-elim.

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{ Property of $\backslash$: }

$y \in F(S) \wedge \neg(y \in F(T))$

{ Property of image: }

$\exists_x [x \in S : F(x) = y]$

Pick an x with $x \in S$ and $F(x) = y$

$\exists_x [x \in S \setminus T : F(x) = y]$

{ Property of image:}

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Example

Let $F : A \to B$ be a mapping, and let $S, T \subseteq A$; then

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To prove that the variable x is a suitable witness for the goal

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To prove that the variable x is a suitable witness for the goal

$$\exists_x[x \in S \setminus T : F(x) = y]$$

we should establish that $x \in S \setminus T$ and $F(x) = y$.

Since $F(x) = y$ is already part of the declaration of x, it remains to prove that $x \in S \setminus T$; this is our new goal.

var y; $y \in F(S) \setminus F(T)$

{ Property of $\setminus$: }

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{ Property of image: }

$\exists_x[x \in S : F(x) = y]$

Pick an x with $x \in S$ and $F(x) = y$

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Since $F(x) = y$ is already part of the declaration of x, it remains to prove that $x \in S \backslash T$; this is our new goal.

The goal $x \in S \backslash T$ can be simplified using the

Property of $\backslash$:
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var y; $y \in F(S) \setminus F(T)$

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{ Property of image: }

$\exists_x [x \in S : F(x) = y]$

Pick an x with $x \in S$ and $F(x) = y$

$x \in S \setminus T$

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Example

Let $F : A \to B$ be a mapping, and let $S, T \subseteq A$; then

$$F(S)\backslash F(T) \subseteq F(S\backslash T) \ .$$

To prove that the variable x is a suitable witness for the goal

$$\exists_x [x \in S\backslash T : F(x) = y]$$

we should establish that $x \in S\backslash T$ and $F(x) = y$.

Since $F(x) = y$ is already part of the declaration of x, it remains to prove that $x \in S\backslash T$; this is our new goal.

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$\exists_x [x \in S : F(x) = y]$

Pick an x with $x \in S$ and $F(x) = y$

$x \in S \wedge \neg(x \in T)$

{ Property of $\backslash$: }

$x \in S \setminus T$

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Let $F : A \to B$ be a mapping, and let $S, T \subseteq A$; then

$$F(S) \setminus F(T) \subseteq F(S \setminus T) \ .$$

To prove that the variable x is a suitable witness for the goal

$$\exists_x[x \in S \setminus T : F(x) = y]$$

we should establish that $x \in S \setminus T$ and $F(x) = y$.

Since $F(x) = y$ is already part of the declaration of x, it remains to prove that $x \in S \setminus T$; this is our new goal.

The goal $x \in S \setminus T$ can be simplified using the

Property of $\setminus$:
$t \in A \setminus B \overset{val}{==} t \in A \wedge \neg(t \in B)$

To prove the conjunction $x \in S \wedge \neg(x \in T)$, we should establish both conjuncts separately. Note that $x \in S$ is already part of the declaration of x, so it remains to establish $\neg(x \in T)$.

$\mathbf{var}\ y;\ y \in F(S) \setminus F(T)$

{ Property of $\setminus$: }

$y \in F(S) \wedge \neg(y \in F(T))$

{ Property of image: }

$\exists_x[x \in S : F(x) = y]$

Pick an x with $x \in S$ and $F(x) = y$

$\neg(x \in T)$

$x \in S \wedge \neg(x \in T)$

{ Property of $\setminus$: }

$x \in S \setminus T$

$\exists_x[x \in S \setminus T : F(x) = y]$

{ Property of image:}

$y \in F(S \setminus T)$

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Example

Let $F : A \rightarrow B$ be a mapping, and let $S, T \subseteq A$; then

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To prove that the variable x is a suitable witness for the goal

$$\exists_x[x \in S \setminus T : F(x) = y]$$

we should establish that $x \in S \setminus T$ and $F(x) = y$.

Since $F(x) = y$ is already part of the declaration of x, it remains to prove that $x \in S \setminus T$; this is our new goal.

The goal $x \in S \setminus T$ can be simplified using the

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$t \in A \setminus B \stackrel{val}{==} t \in A \wedge \neg(t \in B)$

To prove the conjunction $x \in S \wedge \neg(x \in T)$, we should establish both conjuncts separately. Note that $x \in S$ is already part of the declaration of x, so it remains to establish $\neg(x \in T)$.

Example

Let $F : A \to B$ be a mapping, and let $S, T \subseteq A$; then

$$F(S) \backslash F(T) \subseteq F(S \backslash T) \ .$$

Recall that the negation of a formula is proved by assuming that the formula holds, and then deriving a contradiction.

var y; $y \in F(S) \setminus F(T)$

{ Property of $\backslash$: }

$y \in F(S) \wedge \neg(y \in F(T))$

{ Property of image: }

$\exists_x[x \in S : F(x) = y]$

Pick an x with $x \in S$ and $F(x) = y$

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$x \in S \wedge \neg(x \in T)$

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{ Property of image:}

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Recall that the negation of a formula is proved by assuming that the formula holds, and then deriving a contradiction.

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{ Property of $\backslash$: }

$y \in F(S) \wedge \neg(y \in F(T))$

{ Property of image: }

$\exists_x [x \in S : F(x) = y]$

Pick an x with $x \in S$ and $F(x) = y$

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Since $F(x) = y$, it follows, by an application of one of the properties of $=$, that $y \in F(T)$.

Note that $y \in F(T)$ is in contradiction with the second conjunct of the formula on the second line of the proof, and thus the proof is complete.

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We shall now replay the construction of the derivation on the left, and simultaneously construct a proof in natural language.

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Note that, from $y \in F(S) \setminus F(T)$ it follows, by the property of $\setminus$, that $y \in F(S)$ and $y \notin F(T)$. By the property of image, there exists $x \in S$ such that $F(x) = y$.

Recall that we need to prove that $x \in S \setminus T$ and $F(x) = y$. We already have $F(x) = y$, and to prove that $x \in S \setminus T$, by the property of $\setminus$, it remains to prove that $x \notin T$. So suppose that $x \in T$; we derive a contradiction.

From $x \in T$ it follows, by the property of image, that $F(x) \in F(T)$. Hence, since $F(x) = y$, we have that $y \in F(T)$.

We now have $y \in F(T)$ and $y \notin F(T)$: a contradiction. □