Exercise 15.8(a)

Show with a *derivation* that the formula

$$\exists_x \forall_y [P(x, y)] \Rightarrow \forall_v \exists_u [P(u, v)]$$

is a tautology.

Exercise 15.8(a)

Show with a *derivation* that the formula

$$\exists_x \forall_y [P(x, y)] \Rightarrow \forall_v \exists_u [P(u, v)]$$

is a tautology.

Recall that *derivation* is a special kind of formal proof, constructed according to the rules listed in the Tables for Part II on pp. 375–380 of the book.

Exercise 15.8(a)

Show with a *derivation* that the formula

$$\exists_x \forall_y [P(x, y)] \Rightarrow \forall_v \exists_u [P(u, v)]$$

is a tautology.

Recall that *derivation* is a special kind of formal proof, constructed according to the rules listed in the Tables for Part II on pp. 375–380 of the book.

The construction of a derivation starts with writing the goal (i.e., the formula that is to be proved) at the bottom. (Of course, in reality, we do not know in advance that the proof will take 7 lines.)

(1)

(2)

(3)

(4)

(5)

(6)

(7) $\exists_x \forall_y [P(x, y)] \Rightarrow \forall_v \exists_u [P(u, v)]$

Exercise 15.8(a)

Show with a *derivation* that the formula

$$\exists_x \forall_y [P(x,y)] \Rightarrow \forall_v \exists_u [P(u,v)]$$

is a tautology.

Recall that *derivation* is a special kind of formal proof, constructed according to the rules listed in the Tables for Part II on pp. 375–380 of the book.

The construction of a derivation starts with writing the goal (i.e., the formula that is to be proved) at the bottom. (Of course, in reality, we do not know in advance that the proof will take 7 lines.)

The main strategy is then to first try and simplify the goal until it becomes (logically) simple. Preferably, we simplify until our formula has no connectives or quantifiers, but, due to the nature of the rules for $\vee$ and $\exists$, simplification sometimes stops with a formula that has $\vee$ or $\exists$ as main symbol.

(1)

(2)

(3)

(4)

(5)

(6)

(7) $\exists_x \forall_y [P(x,y)] \Rightarrow \forall_v \exists_u [P(u,v)]$

Exercise 15.8(a)

Show with a *derivation* that the formula

$$\exists_x \forall_y [P(x, y)] \Rightarrow \forall_v \exists_u [P(u, v)]$$

is a tautology.

(1)

(2)

(3)

(4)

(5)

(6)

(7) $\exists_x \forall_y [P(x, y)] \Rightarrow \forall_v \exists_u [P(u, v)]$

Exercise 15.8(a)

Show with a *derivation* that the formula

$$\exists_x \forall_y [P(x,y)] \Rightarrow \forall_v \exists_u [P(u,v)]$$

is a tautology.

Note that the derivation rules presented in Part II of the book may not be applied to subformulas. So the main symbol of the formula dictates which rule(s) can be applied. The main symbol in the present goal is $\Rightarrow$.

(1)

(2)

(3)

(4)

(5)

(6)

(7) $\exists_x \forall_y [P(x,y)] \Rightarrow \forall_v \exists_u [P(u,v)]$

Exercise 15.8(a)

Show with a *derivation* that the formula

$$\exists_x \forall_y [P(x, y)] \Rightarrow \forall_v \exists_u [P(u, v)]$$

is a tautology.

Note that the derivation rules presented in Part II of the book may not be applied to subformulas. So the main symbol of the formula dictates which rule(s) can be applied. The main symbol in the present goal is $\Rightarrow$.

The only rule for $\Rightarrow$ (when it is the main symbol of a goal), is the following:

(1)

(2)

(3)

(4)

(5)

(6)

(7) $\exists_x \forall_y [P(x, y)] \Rightarrow \forall_v \exists_u [P(u, v)]$

Exercise 15.8(a)

Show with a *derivation* that the formula

$$\exists_x \forall_y [P(x,y)] \Rightarrow \forall_v \exists_u [P(u,v)]$$

is a tautology.

Note that the derivation rules presented in Part II of the book may not be applied to subformulas. So the main symbol of the formula dictates which rule(s) can be applied. The main symbol in the present goal is $\Rightarrow$.

The only rule for $\Rightarrow$ (when it is the main symbol of a goal), is the following:

$\Rightarrow$-introduction:

Assume:

(k) P

⋮

$(\ell - 1)$ Q

{ $\Rightarrow$-intro on (k) and $(\ell - 1)$: }

(ℓ) $P \Rightarrow Q$

(1)

(2)

(3)

(4)

(5)

(6)

(7) $\exists_x \forall_y [P(x,y)] \Rightarrow \forall_v \exists_u [P(u,v)]$

Exercise 15.8(a)

Show with a *derivation* that the formula

$$\exists_x \forall_y [P(x, y)] \Rightarrow \forall_v \exists_u [P(u, v)]$$

is a tautology.

Note that the derivation rules presented in Part II of the book may not be applied to subformulas. So the main symbol of the formula dictates which rule(s) can be applied. The main symbol in the present goal is $\Rightarrow$.

The only rule for $\Rightarrow$ (when it is the main symbol of a goal), is the following:

$\Rightarrow$-introduction:

Assume:

(k) P

$\vdots$

$(\ell - 1)$ Q

{ $\Rightarrow$-intro on (k) and $(\ell - 1)$: }

(ℓ) $P \Rightarrow Q$

We can apply this rule to the goal by substituting $\exists_x \forall_y [P(x, y)]$ for P and $\forall_v \exists_u [P(u, v)]$ for Q in the rule.

(1)

(2)

(3)

(4)

(5)

(6)

(7) $\exists_x \forall_y [P(x, y)] \Rightarrow \forall_v \exists_u [P(u, v)]$

Exercise 15.8(a)

Show with a *derivation* that the formula

$$\exists_x \forall_y [P(x,y)] \Rightarrow \forall_v \exists_u [P(u,v)]$$

is a tautology.

Note that the derivation rules presented in Part II of the book may not be applied to subformulas. So the main symbol of the formula dictates which rule(s) can be applied. The main symbol in the present goal is $\Rightarrow$.

The only rule for $\Rightarrow$ (when it is the main symbol of a goal), is the following:

$\Rightarrow$-introduction:

Assume:

(k) P

$\vdots$

$(\ell - 1)$ Q

{ $\Rightarrow$-intro on (k) and $(\ell - 1)$: }

(ℓ) $P \Rightarrow Q$

We can apply this rule to the goal by substituting $\exists_x \forall_y [P(x,y)]$ for P and $\forall_v \exists_u [P(u,v)]$ for Q in the rule.

{Assume:}

(1) $\exists_x \forall_y [P(x,y)]$

(2)

(3)

(4)

(5)

(6) $\forall_v [\exists_u [P(u,v)]]$

{ $\Rightarrow$-intro on (1) and (6): }

(7) $\exists_x \forall_y [P(x,y)] \Rightarrow \forall_v \exists_u [P(u,v)]$

Exercise 15.8(a)

Show with a *derivation* that the formula

$$\exists_x \forall_y [P(x,y)] \Rightarrow \forall_v \exists_u [P(u,v)]$$

is a tautology.

{Assume:}

(1) $\exists_x \forall_y [P(x,y)]$

(2)

(3)

(4)

(5)

(6) $\forall_v [\exists_u [P(u,v)]]$

{ $\Rightarrow$-intro on (1) and (6): }

(7) $\exists_x \forall_y [P(x,y)] \Rightarrow \forall_v \exists_u [P(u,v)]$

Exercise 15.8(a)

Show with a *derivation* that the formula

$$\exists_x \forall_y [P(x, y)] \Rightarrow \forall_v \exists_u [P(u, v)]$$

is a tautology.

The original goal has now been simplified; the new goal is $\forall_v [\exists_u [P(u, v)]]$.

{Assume:}

(1) $\exists_x \forall_y [P(x, y)]$

(2)

(3)

(4)

(5)

(6) $\forall_v [\exists_u [P(u, v)]]$

{ $\Rightarrow$-intro on (1) and (6): }

(7) $\exists_x \forall_y [P(x, y)] \Rightarrow \forall_v \exists_u [P(u, v)]$

Exercise 15.8(a)

Show with a *derivation* that the formula

$$\exists_x \forall_y [P(x, y)] \Rightarrow \forall_v \exists_u [P(u, v)]$$

is a tautology.

The original goal has now been simplified; the new goal is $\forall_v [\exists_u [P(u, v)]]$.

The main symbol in the new goal is $\forall$; the only rule for $\forall$ (when it is the main symbol of a goal), is the following:

{Assume:}

(1) $\exists_x \forall_y [P(x, y)]$

(2)

(3)

(4)

(5)

(6) $\forall_v [\exists_u [P(u, v)]]$

{ $\Rightarrow$-intro on (1) and (6): }

(7) $\exists_x \forall_y [P(x, y)] \Rightarrow \forall_v \exists_u [P(u, v)]$

Exercise 15.8(a)

Show with a *derivation* that the formula

$$\exists_x \forall_y [P(x, y)] \Rightarrow \forall_v \exists_u [P(u, v)]$$

is a tautology.

{Assume:}

(1) $\exists_x \forall_y [P(x, y)]$

(2)

(3)

(4)

(5)

(6) $\forall_v [\exists_u [P(u, v)]]$

{ ⇒-intro on (1) and (6): }

(7) $\exists_x \forall_y [P(x, y)] \Rightarrow \forall_v \exists_u [P(u, v)]$

The original goal has now been simplified; the new goal is $\forall_v [\exists_u [P(u, v)]]$.

The main symbol in the new goal is ∀; the only rule for ∀ (when it is the main symbol of a goal), is the following:

∀-introduction:

{ Assume: }

(k) **var** x; $P(x)$

⋮

($\ell - 1$) $Q(x)$

{ ∀-intro on (k) and ($\ell - 1$): }

(ℓ) $\forall_x [P(x) : Q(x)]$

Exercise 15.8(a)

Show with a *derivation* that the formula

$$\exists_x \forall_y [P(x,y)] \Rightarrow \forall_v \exists_u [P(u,v)]$$

is a tautology.

{Assume:}

(1) $\exists_x \forall_y [P(x,y)]$

(2)

(3)

(4)

(5)

(6) $\forall_v [\exists_u [P(u,v)]]$

{ $\Rightarrow$-intro on (1) and (6): }

(7) $\exists_x \forall_y [P(x,y)] \Rightarrow \forall_v \exists_u [P(u,v)]$

The original goal has now been simplified; the new goal is $\forall_v [\exists_u [P(u,v)]]$.

The main symbol in the new goal is $\forall$; the only rule for $\forall$ (when it is the main symbol of a goal), is the following:

$\forall$-introduction:

{ Assume: }

(k) **var** x; $P(x)$

⋮

($\ell - 1$) $Q(x)$

{ $\forall$-intro on (k) and ($\ell - 1$): }

(ℓ) $\forall_x [P(x) : Q(x)]$

In order to apply the rule, we instantiate it with v for x, `True` for $P(x)$ and $\exists_u [P(u,v)]$ for $Q(x)$.

Exercise 15.8(a)

Show with a *derivation* that the formula

$$\exists_x \forall_y [P(x,y)] \Rightarrow \forall_v \exists_u [P(u,v)]$$

is a tautology.

{Assume:}

(1) $\exists_x \forall_y [P(x,y)]$

{ Assume: }

(2) **var** v; `True`

(3)

(4)

(5) $\exists_u [P(u,v)]$

{ $\forall$-intro on (2) and (5): }

(6) $\forall_v [\exists_u [P(u,v)]]$

{ $\Rightarrow$-intro on (1) and (6): }

(7) $\exists_x \forall_y [P(x,y)] \Rightarrow \forall_v \exists_u [P(u,v)]$

The original goal has now been simplified; the new goal is $\forall_v [\exists_u [P(u,v)]]$.

The main symbol in the new goal is $\forall$; the only rule for $\forall$ (when it is the main symbol of a goal), is the following:

$\forall$-introduction:

{ Assume: }

(k) **var** x; $P(x)$

⋮

($\ell - 1$) $Q(x)$

{ $\forall$-intro on (k) and ($\ell - 1$): }

(ℓ) $\forall_x [P(x) : Q(x)]$

In order to apply the rule, we instantiate it with v for x, `True` for $P(x)$ and $\exists_u [P(u,v)]$ for $Q(x)$.

Exercise 15.8(a)

Show with a *derivation* that the formula

$$\exists_x \forall_y [P(x, y)] \Rightarrow \forall_v \exists_u [P(u, v)]$$

is a tautology.

The new goal is $\exists_u [P(u, v)]$, with $\exists$ as main symbol.

{Assume:}

(1) $\exists_x \forall_y [P(x, y)]$

{ Assume: }

(2) **var** v; `True`

(3)

(4)

(5) $\exists_u [P(u, v)]$

{ $\forall$-intro on (2) and (5): }

(6) $\forall_v [\exists_u [P(u, v)]]$

{ $\Rightarrow$-intro on (1) and (6): }

(7) $\exists_x \forall_y [P(x, y)] \Rightarrow \forall_v \exists_u [P(u, v)]$

Exercise 15.8(a)

Show with a *derivation* that the formula

$$\exists_x \forall_y [P(x, y)] \Rightarrow \forall_v \exists_u [P(u, v)]$$

is a tautology.

The new goal is $\exists_u [P(u, v)]$, with $\exists$ as main symbol.

We have two rules suitable for establishing a goal with $\exists$ as main symbol:

{Assume:}

(1) $\exists_x \forall_y [P(x, y)]$

{ Assume: }

(2) **var** v; `True`

(3)

(4)

(5) $\exists_u [P(u, v)]$

{ $\forall$-intro on (2) and (5): }

(6) $\forall_v [\exists_u [P(u, v)]]$

{ $\Rightarrow$-intro on (1) and (6): }

(7) $\exists_x \forall_y [P(x, y)] \Rightarrow \forall_v \exists_u [P(u, v)]$

Exercise 15.8(a)

Show with a *derivation* that the formula

$$\exists_x \forall_y [P(x,y)] \Rightarrow \forall_v \exists_u [P(u,v)]$$

is a tautology.

{Assume:}

(1) $\exists_x \forall_y [P(x,y)]$

{ Assume: }

(2) **var** v; `True`

(3)

(4)

(5) $\exists_u [P(u,v)]$

{ $\forall$-intro on (2) and (5): }

(6) $\forall_v [\exists_u [P(u,v)]]$

{ $\Rightarrow$-intro on (1) and (6): }

(7) $\exists_x \forall_y [P(x,y)] \Rightarrow \forall_v \exists_u [P(u,v)]$

The new goal is $\exists_u [P(u,v)]$, with $\exists$ as main symbol.

We have two rules suitable for establishing a goal with $\exists$ as main symbol:

$\exists$-introduction:

{Assume:}

(k) $\forall_x [P(x) : \neg Q(x)]$

$\vdots$

$(\ell - 1)$ `False`

{ $\exists$-intro on (k) and $(\ell - 1)$: }

(ℓ) $\exists_x [P(x) : Q(x)]$

$\exists^*$-introduction:

$\vdots$

(k) $P(a)$

$\vdots$

(ℓ) $Q(a)$

{ $\exists^*$-intro on (k) and (ℓ): }

(m) $\exists_x [P(x) : Q(x)]$

Exercise 15.8(a)

Show with a *derivation* that the formula

$$\exists_x \forall_y [P(x,y)] \Rightarrow \forall_v \exists_u [P(u,v)]$$

is a tautology.

{Assume:}

(1) $\exists_x \forall_y [P(x,y)]$

{ Assume: }

(2) **var** v; `True`

(3)

(4)

(5) $\exists_u [P(u,v)]$

{ $\forall$-intro on (2) and (5): }

(6) $\forall_v [\exists_u [P(u,v)]]$

{ $\Rightarrow$-intro on (1) and (6): }

(7) $\exists_x \forall_y [P(x,y)] \Rightarrow \forall_v \exists_u [P(u,v)]$

The new goal is $\exists_u [P(u,v)]$, with $\exists$ as main symbol.

We have two rules suitable for establishing a goal with $\exists$ as main symbol:

$\exists$-introduction:

{Assume:}

(k) $\forall_x [P(x) : \neg Q(x)]$

⋮

($\ell - 1$) `False`

{ $\exists$-intro on (k) and ($\ell - 1$): }

(ℓ) $\exists_x [P(x) : Q(x)]$

$\exists^*$-introduction:

⋮

(k) $P(a)$

⋮

(ℓ) $Q(a)$

{ $\exists^*$-intro on (k) and (ℓ): }

(m) $\exists_x [P(x) : Q(x)]$

If there is a complete derivation of the formula at all, then there is one that now proceeds using the left-hand side rule. But derivations using this rule are often longer than derivations using the right-hand side rule instead. Therefore, we recommend to (first) try to find a derivation using the right-hand side rule, and, only if that fails, to backtrack and apply the left-hand side rule.

Exercise 15.8(a)

Show with a *derivation* that the formula

$$\exists_x \forall_y [P(x,y)] \Rightarrow \forall_v \exists_u [P(u,v)]$$

is a tautology.

{Assume:}

(1) $\exists_x \forall_y [P(x,y)]$

{ Assume: }

(2) **var** v; `True`

(3)

(4)

(5) $\exists_u [P(u,v)]$

{ $\forall$-intro on (2) and (5): }

(6) $\forall_v [\exists_u [P(u,v)]]$

{ $\Rightarrow$-intro on (1) and (6): }

(7) $\exists_x \forall_y [P(x,y)] \Rightarrow \forall_v \exists_u [P(u,v)]$

The new goal is $\exists_u [P(u,v)]$, with $\exists$ as main symbol.

We have two rules suitable for establishing a goal with $\exists$ as main symbol:

$\exists$-introduction:

{Assume:}

(k) $\forall_x [P(x) : \neg Q(x)]$

⋮

($\ell - 1$) `False`

{ $\exists$-intro on (k) and ($\ell - 1$): }

(ℓ) $\exists_x [P(x) : Q(x)]$

$\exists^*$-introduction:

⋮

(k) $P(a)$

⋮

(ℓ) $Q(a)$

{ $\exists^*$-intro on (k) and (ℓ): }

(m) $\exists_x [P(x) : Q(x)]$

Exercise 15.8(a)

Show with a *derivation* that the formula

$$\exists_x \forall_y [P(x, y)] \Rightarrow \forall_v \exists_u [P(u, v)]$$

is a tautology.

{Assume:}

(1) $\exists_x \forall_y [P(x, y)]$

{ Assume: }

(2) **var** v; `True`

(3)

(4)

(5) $\exists_u [P(u, v)]$

{ $\forall$-intro on (2) and (5): }

(6) $\forall_v [\exists_u [P(u, v)]]$

{ $\Rightarrow$-intro on (1) and (6): }

(7) $\exists_x \forall_y [P(x, y)] \Rightarrow \forall_v \exists_u [P(u, v)]$

The new goal is $\exists_u [P(u, v)]$, with $\exists$ as main symbol.

We have two rules suitable for establishing a goal with $\exists$ as main symbol:

$\exists$-introduction:

{Assume:}

(k) $\forall_x [P(x) : \neg Q(x)]$

⋮

($\ell - 1$) `False`

{ $\exists$-intro on (k) and ($\ell - 1$): }

(ℓ) $\exists_x [P(x) : Q(x)]$

$\exists^*$-introduction:

⋮

(k) $P(a)$

⋮

(ℓ) $Q(a)$

{ $\exists^*$-intro on (k) and (ℓ): }

(m) $\exists_x [P(x) : Q(x)]$

Note that an application of the right-hand side rule requires us to establish $P(a, v)$ for some suitable object a, which has to be an object that exists in our derivation. Since it is not yet clear what this object a is going to be, let us first turn to forward reasoning: we try to combine the hypotheses to see whether a suitable object a presents itself.

Exercise 15.8(a)

Show with a *derivation* that the formula

$$\exists_x \forall_y [P(x,y)] \Rightarrow \forall_v \exists_u [P(u,v)]$$

is a tautology.

{Assume:}

(1) $\exists_x \forall_y [P(x,y)]$

{ Assume: }

(2) **var** v; `True`

(3)

(4)

(5) $\exists_u [P(u,v)]$

{ $\forall$-intro on (2) and (5): }

(6) $\forall_v [\exists_u [P(u,v)]]$

{ $\Rightarrow$-intro on (1) and (6): }

(7) $\exists_x \forall_y [P(x,y)] \Rightarrow \forall_v \exists_u [P(u,v)]$

Exercise 15.8(a)

Show with a *derivation* that the formula

$$\exists_x \forall_y [P(x, y)] \Rightarrow \forall_v \exists_u [P(u, v)]$$

is a tautology.

Consider the hypothesis in line (1): it has $\exists$ as its main symbol. There are two rules, formalising two different methods, for using the hypothesis:

{Assume:}

(1) $\exists_x \forall_y [P(x, y)]$

{ Assume: }

(2) **var** v; True

(3)

(4)

(5) $\exists_u [P(u, v)]$

{ $\forall$-intro on (2) and (5): }

(6) $\forall_v [\exists_u [P(u, v)]]$

{ $\Rightarrow$-intro on (1) and (6): }

(7) $\exists_x \forall_y [P(x, y)] \Rightarrow \forall_v \exists_u [P(u, v)]$

Exercise 15.8(a)

Show with a *derivation* that the formula

$$\exists_x \forall_y [P(x,y)] \Rightarrow \forall_v \exists_u [P(u,v)]$$

is a tautology.

{Assume:}

(1) $\exists_x \forall_y [P(x,y)]$

{ Assume: }

(2) **var** v; `True`

(3)

(4)

(5) $\exists_u [P(u,v)]$

{ $\forall$-intro on (2) and (5): }

(6) $\forall_v [\exists_u [P(u,v)]]$

{ $\Rightarrow$-intro on (1) and (6): }

(7) $\exists_x \forall_y [P(x,y)] \Rightarrow \forall_v \exists_u [P(u,v)]$

Consider the hypothesis in line (1): it has $\exists$ as its main symbol. There are two rules, formalising two different methods, for using the hypothesis:

$\exists$-elimination:

$\vdots$

(k) $\exists_x [P(x) : Q(x)]$

$\vdots$

(ℓ) $\forall_x [P(x) : \neg Q(x)]$

$\vdots$

{ $\exists$-elim on (k) and (ℓ):}

(m) `False`

$\exists^*$-elimination:

$\vdots$

(k) $\exists_x [P(x) : Q(x)]$

$\vdots$

{$\exists^*$-elim on (k):}

(ℓ) Pick an x with $P(x)$ and $Q(x)$

Exercise 15.8(a)

Show with a *derivation* that the formula

$$\exists_x \forall_y [P(x,y)] \Rightarrow \forall_v \exists_u [P(u,v)]$$

is a tautology.

{Assume:}

(1) $\exists_x \forall_y [P(x,y)]$

{ Assume: }

(2) **var** v; `True`

(3)

(4)

(5) $\exists_u [P(u,v)]$

{ $\forall$-intro on (2) and (5): }

(6) $\forall_v [\exists_u [P(u,v)]]$

{ $\Rightarrow$-intro on (1) and (6): }

(7) $\exists_x \forall_y [P(x,y)] \Rightarrow \forall_v \exists_u [P(u,v)]$

Consider the hypothesis in line (1): it has $\exists$ as its main symbol. There are two rules, formalising two different methods, for using the hypothesis:

$\exists$-elimination:

|||

(k) $\exists_x [P(x) : Q(x)]$

|||

(ℓ) $\forall_x [P(x) : \neg Q(x)]$

|||

{ $\exists$-elim on (k) and (ℓ):}

(m) `False`

$\exists^*$-elimination:

|||

(k) $\exists_x [P(x) : Q(x)]$

|||

{$\exists^*$-elim on (k):}

(ℓ) Pick an x with $P(x)$ and $Q(x)$

Since we do not (also) have $\forall_x [\neg \forall_y [P(x,y)]]$, we cannot directly apply the left-hand side rule. If we were out of options, we would now introduce this formula as a new subgoal in our derivation. But actually we are not: we can directly apply the right-hand side rule.

Exercise 15.8(a)

Show with a *derivation* that the formula

$$\exists_x \forall_y [P(x,y)] \Rightarrow \forall_v \exists_u [P(u,v)]$$

is a tautology.

{Assume:}

(1) $\exists_x \forall_y [P(x,y)]$

{ Assume: }

(2) **var** v; `True`

(3)

(4)

(5) $\exists_u [P(u,v)]$

{ $\forall$-intro on (2) and (5): }

(6) $\forall_v [\exists_u [P(u,v)]]$

{ $\Rightarrow$-intro on (1) and (6): }

(7) $\exists_x \forall_y [P(x,y)] \Rightarrow \forall_v \exists_u [P(u,v)]$

Consider the hypothesis in line (1): it has $\exists$ as its main symbol. There are two rules, formalising two different methods, for using the hypothesis:

$\exists$-elimination:

	$\Vert\vert$
(k)	$\exists_x [P(x) : Q(x)]$
	$\Vert\vert$
(ℓ)	$\forall_x [P(x) : \neg Q(x)]$
	$\Vert\vert$
	{ $\exists$-elim on (k) and (ℓ):}
(m)	`False`

$\exists^*$-elimination:

	$\Vert\vert$
(k)	$\exists_x [P(x) : Q(x)]$
	$\Vert\vert$
	{$\exists^*$-elim on (k):}
(ℓ)	Pick an x with $P(x)$ and $Q(x)$

Since we do not (also) have $\forall_x [\neg \forall_y [P(x,y)]]$, we cannot directly apply the left-hand side rule. If we were out of options, we would now introduce this formula as a new subgoal in our derivation. But actually we are not: we can directly apply the right-hand side rule. To apply it, we instantiate it with x for x, `True` for $P(x)$, and $\forall_y [P(x,y)]$ for $Q(x)$. (Note that `True` and has been omitted from line (3).)

Exercise 15.8(a)

Show with a *derivation* that the formula

$$\exists_x \forall_y [P(x,y)] \Rightarrow \forall_v \exists_u [P(u,v)]$$

is a tautology.

{Assume:}

(1) $\exists_x \forall_y [P(x,y)]$

{ Assume: }

(2) **var** v; `True`

{ $\exists^*$-elim on (1): }

(3) Pick an x with $\forall_y [P(x,y)]$

(4)

(5) $\exists_u [P(u,v)]$

{ $\forall$-intro on (2) and (5): }

(6) $\forall_v [\exists_u [P(u,v)]]$

{ $\Rightarrow$-intro on (1) and (6): }

(7) $\exists_x \forall_y [P(x,y)] \Rightarrow \forall_v \exists_u [P(u,v)]$

Consider the hypothesis in line (1): it has $\exists$ as its main symbol. There are two rules, formalising two different methods, for using the hypothesis:

$\exists$-elimination:

	$\Vert\Vert$
(k)	$\exists_x [P(x) : Q(x)]$
	$\Vert\Vert$
(ℓ)	$\forall_x [P(x) : \neg Q(x)]$
	$\Vert\Vert$
	{ $\exists$-elim on (k) and (ℓ):}
(m)	`False`

$\exists^*$-elimination:

	$\Vert\Vert$
(k)	$\exists_x [P(x) : Q(x)]$
	$\Vert\Vert$
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Show with a *derivation* that the formula

$$\exists_x \forall_y [P(x,y)] \Rightarrow \forall_v \exists_u [P(u,v)]$$

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(2) **var** v; `True`

{ $\exists^*$-elim on (1): }

(3) Pick an x with $\forall_y [P(x,y)]$

(4)

(5) $\exists_u [P(u,v)]$

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We can now establish $P(x, v)$ by means of an application of $\forall$-elim on (3), which then conveniently allows us to conclude our current goal $\exists_u[P(u, v)]$ with an application of $\exists^*$-intro (as already announced). Let's work this out in detail:

{Assume:}

(1) $\exists_x \forall_y [P(x, y)]$

{ Assume: }

(2) **var** v; `True`

{ $\exists^*$-elim on (1): }

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We start with the application of $\forall$-elim:

$\forall$-elimination:

$\|\|$

(k) $\forall_x [P(x) : Q(x)]$

$\|\|$

(ℓ) $P(a)$

$\|\|$

{$\forall$-elim on (k) and (ℓ):}

(m) $Q(a)$

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$\forall$-elimination:

	⋮
(k)	$\forall_x [P(x) : Q(x)]$
	⋮
(ℓ)	$P(a)$
	⋮
	{$\forall$-elim on (k) and (ℓ):}
(m)	$Q(a)$

To apply it, we instantiate it with y for x, `True` for $P(x)$, and $P(x,y)$ for $Q(x)$, and v for a. It is of crucial importance for a correct application of the rule that the object v, which is used to instantiate a, is 'known' in line (3); indeed it is declared in line (2), which is still part of the context.

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(4) $P(x,v)$

(5) $\exists_u [P(u,v)]$

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	$\Vert\Vert$
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	$\Vert\Vert$
(ℓ)	$P(a)$
	$\Vert\Vert$
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(5) $\exists_u [P(u, v)]$

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{ Assume: }

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Then, we can complete the derivation with an application of $\exists^*$-elim:

$\exists^*$-introduction:

$\vdots$

(k) $P(a)$

$\vdots$

(ℓ) $Q(a)$

{ $\exists^*$-intro on (k) and (ℓ): }

(m) $\exists_x [P(x) : Q(x)]$

Exercise 15.8(a)

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{ Assume: }

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{ $\exists^*$-elim on (1): }

(3) Pick an x with $\forall_y [P(x,y)]$

{ $\forall$-elim on (3) and (2): }

(4) $P(x,v)$

(5) $\exists_u [P(u,v)]$

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$\exists^*$-introduction:

⋮

(k) $P(a)$

⋮

(ℓ) $Q(a)$

{ $\exists^*$-intro on (k) and (ℓ): }

(m) $\exists_x [P(x) : Q(x)]$

To apply it, we instantiate it with u for x, `True` for $P(x)$, and $P(u, v)$ for $Q(x)$.

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{ $\forall$-elim on (3) and (2): }

(4) $P(x, v)$

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We can now establish $P(x, v)$ by means of an application of $\forall$-elim on (3), which then conveniently allows us to conclude our current goal $\exists_u [P(u, v)]$ with an application of $\exists^*$-intro (as already announced). Let's work this out in detail:

Then, we can complete the derivation with an application of $\exists^*$-elim:

$\exists^*$-introduction:

$\vdots$

(k) $P(a)$

$\vdots$

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{ $\exists^*$-intro on (k) and (ℓ): }

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To apply it, we instantiate it with u for x, `True` for $P(x)$, and $P(u, v)$ for $Q(x)$.

Exercise 15.8(a)

Show with a *derivation* that the formula

$$\exists_x \forall_y [P(x, y)] \Rightarrow \forall_v \exists_u [P(u, v)]$$

is a tautology.

{Assume:}

(1) $\boxed{\exists_x \forall_y [P(x,y)]}$

{ Assume: }

(2) $\boxed{\textbf{var } v;\ \texttt{True}}$

{ $\exists^*$-elim on (1): }

(3) Pick an x with $\forall_y [P(x,y)]$

{ $\forall$-elim on (3) and (2): }

(4) $P(x, v)$

{ $\exists^*$-intro on (4): }

(5) $\exists_u [P(u, v)]$

{ $\forall$-intro on (2) and (5): }

(6) $\forall_v [\exists_u [P(u, v)]]$

{ $\Rightarrow$-intro on (1) and (6): }

(7) $\exists_x \forall_y [P(x,y)] \Rightarrow \forall_v \exists_u [P(u,v)]$

The derivation is now complete; we conclude that the formula $\exists_x \forall_y [P(x, y) \Rightarrow \forall_v \exists_u [P(u, v)]$ is a tautology.