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We give a *derivation-style* proof of this property.

The goal (the formula to be proved) is an implication, so we apply the rule for $\Rightarrow$-intro to simplify the goal.

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{ Assume: }

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The goal (the formula to be proved) is an implication, so we apply the rule for $\Rightarrow$-intro to simplify the goal.

For proving the new goal $B \setminus A = \emptyset$ we have several methods at our disposal (see the table on p. 381 of the book):

1. we can use the definition of $=$ in terms of the $\subseteq$-predicate:

   $$\boxed{A = B \stackrel{\text{def}}{=} A \subseteq B \wedge B \subseteq A}$$

2. we can use the (derived)

   Property of $=$:
   $A = B \stackrel{val}{==} \forall_x[x \in A \Leftrightarrow x \in B]$

3. we can use the (derived)

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{ Assume: }

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Each of these methods can be applied, and may (eventually) lead to a proof. The latter method seems the most appropriate here, because it is most specific.

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{ Prop. of $\emptyset$ on (10): }

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{ Assume: }

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(4)

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(8)

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{ $\forall$-intro on (2) and (9): }

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The goal is now to derive a contradiction ($\texttt{False}$) from the valid assumptions $A \cup B = A$ and $x \in B \setminus A$.

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Let's first infer two simple facts $x \in B$ and $\neg(x \in A)$ from the assumption $x \in B \setminus A$ first, applying

Property of $\setminus$:

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Recall that our goal is to derive a contradiction (`False`). Most of the time (but not always), this is done by establishing both some formula P and its negation $\neg P$, and then conclude `False` with an application of $\neg$-elim.

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{ $\neg$-elim on (8) and (5): }

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To obtain the new goal $x \in A$, it is to be expected that we need the assumption $A \cup B = A$ (since it has not yet been used) and, perhaps, also $x \in B$.

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(6)

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{ $\forall$-intro on (2) and (9): }

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Intuitively, if $x \in B$, then also $x \in A \cup B$, and since $A \cup B = A$, and hence it follows that $x \in A$. We can formalise this informal reasoning thus:

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{ Assume: }

(2) **var** x; $x \in B \setminus A$

{ Prop. of $\setminus$ on (2): }

(3) $x \in B \wedge \neg(x \in A)$

{ $\wedge$-elim on (3):}

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(6)

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Note that, by $\wedge$-$\vee$-weakening, $x \in B \overset{val}{\models} x \in A \vee x \in B$, so $x \in A \vee x \in B$ is a *correct conclusion* from the premisse $x \in B$.

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{ Prop. of $\setminus$ on (2): }

(3) $x \in B \wedge \neg(x \in A)$

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{ $\neg$-elim on (8) and (5): }

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{ $\wedge$-$\vee$-weakening on (4): }

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(7)

(8) $x \in A$

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(11) $B \setminus A = \emptyset$

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{ Assume: }

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(2) **var** x; $x \in B \setminus A$

{ Prop. of $\setminus$ on (2): }

(3) $x \in B \wedge \neg(x \in A)$

{ $\wedge$-elim on (3):}

(4) $x \in B$

{ $\wedge$-elim on (3): }

(5) $\neg(x \in A)$

{ $\wedge$-$\vee$-weakening on (4): }

(6) $x \in A \vee x \in B$

{ Prop. of $\cup$ on (6): }

(7) $x \in A \cup B$

(8) $x \in A$

{ $\neg$-elim on (8) and (5): }

(9) `False`

{ $\forall$-intro on (2) and (9): }

(10) $\forall_x[x \in B \setminus A : \texttt{False}]$

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Let A and B be sets; prove that the following property holds:

$$A \cup B = A \Rightarrow B \setminus A = \emptyset \ .$$

To obtain the new goal $x \in A$, it is to be expected that we need the assumption $A \cup B = A$ (since it has not yet been used) and, perhaps, also $x \in B$.

Intuitively, if $x \in B$, then also $x \in A \cup B$, and since $A \cup B = A$, and hence it follows that $x \in A$. We can formalise this informal reasoning thus:

Note that, by $\wedge$-$\vee$-weakening, $x \in B \overset{val}{\models} x \in A \vee x \in B$, so $x \in A \vee x \in B$ is a *correct conclusion* from the premisse $x \in B$.

On the formula $x \in A \vee x \in B$ we can apply the

Property of $\cup$:
$t \in A \cup B \overset{val}{==} t \in A \vee t \in B$

to obtain $x \in A \cup B$.

And finally, with an application of the

Property of $=$:
$A = B \wedge t \in A \overset{val}{\models} t \in B$

we infer $x \in A$ from the assumption $A \cup B = A$ and $x \in A \cup B$.

{ Assume: }

(1) $A \cup B = A$

{ Assume: }

(2) **var** x; $x \in B \setminus A$

{ Prop. of $\setminus$ on (2): }

(3) $x \in B \wedge \neg(x \in A)$

{ $\wedge$-elim on (3):}

(4) $x \in B$

{ $\wedge$-elim on (3): }

(5) $\neg(x \in A)$

{ $\wedge$-$\vee$-weakening on (4): }

(6) $x \in A \vee x \in B$

{ Prop. of $\cup$ on (6): }

(7) $x \in A \cup B$

(8) $x \in A$

{ $\neg$-elim on (8) and (5): }

(9) False

{ $\forall$-intro on (2) and (9): }

(10) $\forall_x[x \in B \setminus A : \texttt{False}]$

{ Prop. of $\emptyset$ on (10): }

(11) $B \setminus A = \emptyset$

{ $\Rightarrow$-intro on (1) and (11): }

(12) $A \cup B = A \Rightarrow B \setminus A = \emptyset$

Example

Let A and B be sets; prove that the following property holds:

$$A \cup B = A \Rightarrow B \setminus A = \emptyset \ .$$

To obtain the new goal $x \in A$, it is to be expected that we need the assumption $A \cup B = A$ (since it has not yet been used) and, perhaps, also $x \in B$.

Intuitively, if $x \in B$, then also $x \in A \cup B$, and since $A \cup B = A$, and hence it follows that $x \in A$. We can formalise this informal reasoning thus:

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> **Property of $\cup$:**
> $t \in A \cup B \overset{val}{=\!=} t \in A \vee t \in B$

to obtain $x \in A \cup B$.

And finally, with an application of the

> **Property of $=$:**
> $A = B \wedge t \in A \overset{val}{\models} t \in B$

we infer $x \in A$ from the assumption $A \cup B = A$ and $x \in A \cup B$.

{ Assume: }

(1) $A \cup B = A$

{ Assume: }

(2) **var** x; $x \in B \setminus A$

{ Prop. of $\setminus$ on (2): }

(3) $x \in B \wedge \neg(x \in A)$

{ $\wedge$-elim on (3):}

(4) $x \in B$

{ $\wedge$-elim on (3): }

(5) $\neg(x \in A)$

{ $\wedge$-$\vee$-weakening on (4): }

(6) $x \in A \vee x \in B$

{ Prop. of $\cup$ on (6): }

(7) $x \in A \cup B$

{ Property of $=$ on (7) and (1): }

(8) $x \in A$

{ $\neg$-elim on (8) and (5): }

(9) False

{ $\forall$-intro on (2) and (9): }

(10) $\forall_x[x \in B \setminus A : \texttt{False}]$

{ Prop. of $\emptyset$ on (10): }

(11) $B \setminus A = \emptyset$

{ $\Rightarrow$-intro on (1) and (11): }

(12) $A \cup B = A \Rightarrow B \setminus A = \emptyset$

{ Assume: }

(1) $A \cup B = A$

{ Assume: }

(2) **var** x; $x \in B \setminus A$

{ Prop. of $\setminus$ on (2): }

(3) $x \in B \wedge \neg(x \in A)$

{ $\wedge$-elim on (3):}

(4) $x \in B$

{ $\wedge$-elim on (3): }

(5) $\neg(x \in A)$

{ $\wedge$-$\vee$-weakening on (4): }

(6) $x \in A \vee x \in B$

{ Prop. of $\cup$ on (6): }

(7) $x \in A \cup B$

{ Property of $=$ on (7) and (1): }

(8) $x \in A$

{ $\neg$-elim on (8) and (5): }

(9) False

{ $\forall$-intro on (2) and (9): }

(10) $\forall_x[x \in B \setminus A : \texttt{False}]$

{ Prop. of $\emptyset$ on (10): }

(11) $B \setminus A = \emptyset$

{ $\Rightarrow$-intro on (1) and (11): }

(12) $A \cup B = A \Rightarrow B \setminus A = \emptyset$

Example

Let A and B be sets; prove that the following property holds:

$$A \cup B = A \Rightarrow B \setminus A = \emptyset \ .$$

The proof on the left is now complete, and it establishes that

$$A \cup B = A \Rightarrow B \setminus A = \emptyset$$

for all sets A and B.

Example

Let A and B be sets; prove that the following property holds:

$$A \cup B = A \Rightarrow B \setminus A = \emptyset \ .$$

The proof on the left is now complete, and it establishes that

$$A \cup B = A \Rightarrow B \setminus A = \emptyset$$

for all sets A and B.

We shall now incrementally rebuild the derivation, and simultaneously construct a proof in natural language:

Example

Let A and B be sets; prove that the following property holds:

$$A \cup B = A \Rightarrow B \setminus A = \emptyset \ .$$

The proof on the left is now complete, and it establishes that

$$A \cup B = A \Rightarrow B \setminus A = \emptyset$$

for all sets A and B.

We shall now incrementally rebuild the derivation, and simultaneously construct a proof in natural language:

> *To prove the implication* $A \cup B = A \Rightarrow B \setminus A = \emptyset$

(1)

(2)

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(9)

(10)

(11)

(12) $A \cup B = A \Rightarrow B \setminus A = \emptyset$

{ Assume: }

(1) $A \cup B = A$

(2)

(3)

(4)

(5)

(6)

(7)

(8)

(9)

(10)

(11)

{ $\Rightarrow$-intro on (1) and (11): }

(12) $A \cup B = A \Rightarrow B \setminus A = \emptyset$

Example

Let A and B be sets; prove that the following property holds:

$$A \cup B = A \Rightarrow B \setminus A = \emptyset \ .$$

The proof on the left is now complete, and it establishes that

$$A \cup B = A \Rightarrow B \setminus A = \emptyset$$

for all sets A and B.

We shall now incrementally rebuild the derivation, and simultaneously construct a proof in natural language:

> *To prove the implication $A \cup B = A \Rightarrow B \setminus A = \emptyset$, we assume that $A \cup B = A$*

{ Assume: }

(1) $A \cup B = A$

(2)

(3)

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(6)

(7)

(8)

(9)

(10)

(11) $B \setminus A = \emptyset$

{ $\Rightarrow$-intro on (1) and (11): }

(12) $A \cup B = A \Rightarrow B \setminus A = \emptyset$

Example

Let A and B be sets; prove that the following property holds:

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The proof on the left is now complete, and it establishes that

$$A \cup B = A \Rightarrow B \setminus A = \emptyset$$

for all sets A and B.

We shall now incrementally rebuild the derivation, and simultaneously construct a proof in natural language:

> *To prove the implication $A \cup B = A \Rightarrow B \setminus A = \emptyset$, we assume that $A \cup B = A$ and establish that $B \setminus A = \emptyset$.*

{ Assume: }

(1) $A \cup B = A$

(2)

(3)

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(6)

(7)

(8)

(9)

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for all sets A and B.

We shall now incrementally rebuild the derivation, and simultaneously construct a proof in natural language:

> *To prove the implication $A \cup B = A \Rightarrow B \setminus A = \emptyset$, we assume that $A \cup B = A$ and establish that $B \setminus A = \emptyset$.*
>
> *Note that, to establish $B \setminus A = \emptyset$, it suffices*

{ Assume: }

(1) $A \cup B = A$

(2)

(3)

(4)

(5)

(6)

(7)

(8)

(9)

(10) $\forall_x [x \in B \setminus A : \texttt{False}]$

{ Prop. of $\emptyset$ on (10): }

(11) $B \setminus A = \emptyset$

{ $\Rightarrow$-intro on (1) and (11): }

(12) $A \cup B = A \Rightarrow B \setminus A = \emptyset$

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Let A and B be sets; prove that the following property holds:

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>
> *Note that, to establish $B \setminus A = \emptyset$, it suffices, by the Property of $\emptyset$,*

{ Assume: }

(1) $A \cup B = A$

{ Assume: }

(2) **var** x; $x \in B \setminus A$

(3)

(4)

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(8)

(9)

{ $\forall$-intro on (2) and (9): }

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> *To prove the implication $A \cup B = A \Rightarrow B \setminus A = \emptyset$, we assume that $A \cup B = A$ and establish that $B \setminus A = \emptyset$.*
>
> *Note that, to establish $B \setminus A = \emptyset$, it suffices, by the Property of $\emptyset$, to consider an arbitrary element $x \in B \setminus A$*

{ Assume: }

(1) $A \cup B = A$

{ Assume: }

(2) **var** x; $x \in B \setminus A$

(3)

(4)

(5)

(6)

(7)

(8)

(9) `False`

{ $\forall$-intro on (2) and (9): }

(10) $\forall_x [x \in B \setminus A : \texttt{False}]$

{ Prop. of $\emptyset$ on (10): }

(11) $B \setminus A = \emptyset$

{ $\Rightarrow$-intro on (1) and (11): }

(12) $A \cup B = A \Rightarrow B \setminus A = \emptyset$

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> *Note that, to establish $B \setminus A = \emptyset$, it suffices, by the Property of $\emptyset$, to consider an arbitrary element $x \in B \setminus A$ and derive a contradiction.*

Example

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>
> *Note that, to establish $B \setminus A = \emptyset$, it suffices, by the Property of $\emptyset$, to consider an arbitrary element $x \in B \setminus A$ and derive a contradiction.*
>
> *From $x \in B \setminus A$, it follows, by the Property of $\setminus$, that*

{ Assume: }

(1) $A \cup B = A$

  { Assume: }

(2) **var** x; $x \in B \setminus A$

    { Prop. of $\setminus$ on (2): }

(3) $x \in B \wedge \neg(x \in A)$

(4)

(5)

(6)

(7)

(8)

(9) `False`

  { $\forall$-intro on (2) and (9): }

(10) $\forall_x[x \in B \setminus A : \texttt{False}]$

  { Prop. of $\emptyset$ on (10): }

(11) $B \setminus A = \emptyset$

{ $\Rightarrow$-intro on (1) and (11): }

(12) $A \cup B = A \Rightarrow B \setminus A = \emptyset$

	{ Assume: }
(1)	$A \cup B = A$
	{ Assume: }
(2)	**var** x; $x \in B \setminus A$
	{ Prop. of $\setminus$ on (2): }
(3)	$x \in B \wedge \neg(x \in A)$
	{ $\wedge$-elim on (3):}
(4)	$x \in B$
	{ $\wedge$-elim on (3): }
(5)	$\neg(x \in A)$
(6)	
(7)	
(8)	
(9)	`False`
	{ $\forall$-intro on (2) and (9): }
(10)	$\forall_x[x \in B \setminus A : \texttt{False}]$
	{ Prop. of $\emptyset$ on (10): }
(11)	$B \setminus A = \emptyset$
	{ $\Rightarrow$-intro on (1) and (11): }
(12)	$A \cup B = A \Rightarrow B \setminus A = \emptyset$

Example

Let A and B be sets; prove that the following property holds:

$$A \cup B = A \Rightarrow B \setminus A = \emptyset \ .$$

The proof on the left is now complete, and it establishes that

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for all sets A and B.

We shall now incrementally rebuild the derivation, and simultaneously construct a proof in natural language:

> *To prove the implication $A \cup B = A \Rightarrow B \setminus A = \emptyset$, we assume that $A \cup B = A$ and establish that $B \setminus A = \emptyset$.*
>
> *Note that, to establish $B \setminus A = \emptyset$, it suffices, by the Property of $\emptyset$, to consider an arbitrary element $x \in B \setminus A$ and derive a contradiction.*
>
> *From $x \in B \setminus A$, it follows, by the Property of $\setminus$, that $x \in B$ and $\neg(x \in A)$.*

Example

Let A and B be sets; prove that the following property holds:

$$A \cup B = A \Rightarrow B \setminus A = \emptyset \ .$$

The proof on the left is now complete, and it establishes that

$$A \cup B = A \Rightarrow B \setminus A = \emptyset$$

for all sets A and B.

We shall now incrementally rebuild the derivation, and simultaneously construct a proof in natural language:

> *To prove the implication $A \cup B = A \Rightarrow B \setminus A = \emptyset$, we assume that $A \cup B = A$ and establish that $B \setminus A = \emptyset$.*
>
> *Note that, to establish $B \setminus A = \emptyset$, it suffices, by the Property of $\emptyset$, to consider an arbitrary element $x \in B \setminus A$ and derive a contradiction.*
>
> *From $x \in B \setminus A$, it follows, by the Property of $\setminus$, that $x \in B$ and $\neg(x \in A)$.*
>
> *From $x \in B$ it then follows, by the Property of $\cup$, that $x \in A \cup B$*

{ Assume: }

(1) $\boxed{A \cup B = A}$

{ Assume: }

(2) $\boxed{\textbf{var } x;\ x \in B \setminus A}$

{ Prop. of $\setminus$ on (2): }

(3) $x \in B \wedge \neg(x \in A)$

{ $\wedge$-elim on (3):}

(4) $x \in B$

{ $\wedge$-elim on (3): }

(5) $\neg(x \in A)$

{ $\wedge$-$\vee$-weakening on (4): }

(6) $x \in A \vee x \in B$

{ Prop. of $\cup$ on (6): }

(7) $x \in A \cup B$

(8)

(9) `False`

{ $\forall$-intro on (2) and (9): }

(10) $\forall_x[x \in B \setminus A : \texttt{False}]$

{ Prop. of $\emptyset$ on (10): }

(11) $B \setminus A = \emptyset$

{ $\Rightarrow$-intro on (1) and (11): }

(12) $A \cup B = A \Rightarrow B \setminus A = \emptyset$

Example

Let A and B be sets; prove that the following property holds:

$$A \cup B = A \Rightarrow B \setminus A = \emptyset \ .$$

The proof on the left is now complete, and it establishes that

$$A \cup B = A \Rightarrow B \setminus A = \emptyset$$

for all sets A and B.

We shall now incrementally rebuild the derivation, and simultaneously construct a proof in natural language:

> *To prove the implication $A \cup B = A \Rightarrow B \setminus A = \emptyset$, we assume that $A \cup B = A$ and establish that $B \setminus A = \emptyset$.*
>
> *Note that, to establish $B \setminus A = \emptyset$, it suffices, by the Property of $\emptyset$, to consider an arbitrary element $x \in B \setminus A$ and derive a contradiction.*
>
> *From $x \in B \setminus A$, it follows, by the Property of $\setminus$, that $x \in B$ and $\neg(x \in A)$.*
>
> *From $x \in B$ it then follows, by the Property of $\cup$, that $x \in A \cup B$*
>
> *Since $A \cup B = A$, it follows from $x \in A \cup B$, by the Property of $=$, that $x \in A$.*

{ Assume: }

(1) $A \cup B = A$

{ Assume: }

(2) **var** x; $x \in B \setminus A$

{ Prop. of $\setminus$ on (2): }

(3) $x \in B \wedge \neg(x \in A)$

{ $\wedge$-elim on (3):}

(4) $x \in B$

{ $\wedge$-elim on (3): }

(5) $\neg(x \in A)$

{ $\wedge$-$\vee$-weakening on (4): }

(6) $x \in A \vee x \in B$

{ Prop. of $\cup$ on (6): }

(7) $x \in A \cup B$

{ Property of $=$ on (7) and (1): }

(8) $x \in A$

(9) $\texttt{False}$

{ $\forall$-intro on (2) and (9): }

(10) $\forall_x[x \in B \setminus A : \texttt{False}]$

{ Prop. of $\emptyset$ on (10): }

(11) $B \setminus A = \emptyset$

{ $\Rightarrow$-intro on (1) and (11): }

(12) $A \cup B = A \Rightarrow B \setminus A = \emptyset$

{ Assume: }

(1) $A \cup B = A$

{ Assume: }

(2) **var** x; $x \in B \setminus A$

{ Prop. of $\setminus$ on (2): }

(3) $x \in B \wedge \neg(x \in A)$

{ $\wedge$-elim on (3):}

(4) $x \in B$

{ $\wedge$-elim on (3): }

(5) $\neg(x \in A)$

{ $\wedge$-$\vee$-weakening on (4): }

(6) $x \in A \vee x \in B$

{ Prop. of $\cup$ on (6): }

(7) $x \in A \cup B$

{ Property of $=$ on (7) and (1): }

(8) $x \in A$

(9) `False`

{ $\forall$-intro on (2) and (9): }

(10) $\forall_x[x \in B \setminus A : \texttt{False}]$

{ Prop. of $\emptyset$ on (10): }

(11) $B \setminus A = \emptyset$

{ $\Rightarrow$-intro on (1) and (11): }

(12) $A \cup B = A \Rightarrow B \setminus A = \emptyset$

Example

Let A and B be sets; prove that the following property holds:

$$A \cup B = A \Rightarrow B \setminus A = \emptyset \ .$$

The proof on the left is now complete, and it establishes that

$$A \cup B = A \Rightarrow B \setminus A = \emptyset$$

for all sets A and B.

We shall now incrementally rebuild the derivation, and simultaneously construct a proof in natural language:

To prove the implication $A \cup B = A \Rightarrow B \setminus A = \emptyset$, we assume that $A \cup B = A$ and establish that $B \setminus A = \emptyset$.

Note that, to establish $B \setminus A = \emptyset$, it suffices, by the Property of $\emptyset$, to consider an arbitrary element $x \in B \setminus A$ and derive a contradiction.

From $x \in B \setminus A$, it follows, by the Property of $\setminus$, that $x \in B$ and $\neg(x \in A)$.

From $x \in B$ it then follows, by the Property of $\cup$, that $x \in A \cup B$

Since $A \cup B = A$, it follows from $x \in A \cup B$, by the Property of $=$, that $x \in A$.

Note that we have now proved both $\neg(x \in A)$ and $x \in A$,

{ Assume: }

(1) $A \cup B = A$

{ Assume: }

(2) **var** x; $x \in B \setminus A$

{ Prop. of $\setminus$ on (2): }

(3) $x \in B \wedge \neg(x \in A)$

{ $\wedge$-elim on (3):}

(4) $x \in B$

{ $\wedge$-elim on (3): }

(5) $\neg(x \in A)$

{ $\wedge$-$\vee$-weakening on (4): }

(6) $x \in A \vee x \in B$

{ Prop. of $\cup$ on (6): }

(7) $x \in A \cup B$

{ Property of $=$ on (7) and (1): }

(8) $x \in A$

{ $\neg$-elim on (8) and (5): }

(9) `False`

{ $\forall$-intro on (2) and (9): }

(10) $\forall_x[x \in B \setminus A : \texttt{False}]$

{ Prop. of $\emptyset$ on (10): }

(11) $B \setminus A = \emptyset$

{ $\Rightarrow$-intro on (1) and (11): }

(12) $A \cup B = A \Rightarrow B \setminus A = \emptyset$

Example

Let A and B be sets; prove that the following property holds:

$$A \cup B = A \Rightarrow B \setminus A = \emptyset \ .$$

The proof on the left is now complete, and it establishes that

$$A \cup B = A \Rightarrow B \setminus A = \emptyset$$

for all sets A and B.

We shall now incrementally rebuild the derivation, and simultaneously construct a proof in natural language:

To prove the implication $A \cup B = A \Rightarrow B \setminus A = \emptyset$, we assume that $A \cup B = A$ and establish that $B \setminus A = \emptyset$.

Note that, to establish $B \setminus A = \emptyset$, it suffices, by the Property of $\emptyset$, to consider an arbitrary element $x \in B \setminus A$ and derive a contradiction.

From $x \in B \setminus A$, it follows, by the Property of $\setminus$, that $x \in B$ and $\neg(x \in A)$.

From $x \in B$ it then follows, by the Property of $\cup$, that $x \in A \cup B$

Since $A \cup B = A$, it follows from $x \in A \cup B$, by the Property of $=$, that $x \in A$.

Note that we have now proved both $\neg(x \in A)$ and $x \in A$, so we have our contradiction.

Example

Let A and B be sets; prove that the following property holds:

$$A \cup B = A \Rightarrow B \setminus A = \emptyset \ .$$

{ Assume: }

(1) $A \cup B = A$

{ Assume: }

(2) **var** x; $x \in B \setminus A$

{ Prop. of $\setminus$ on (2): }

(3) $x \in B \wedge \neg(x \in A)$

{ $\wedge$-elim on (3):}

(4) $x \in B$

{ $\wedge$-elim on (3): }

(5) $\neg(x \in A)$

{ $\wedge$-$\vee$-weakening on (4): }

(6) $x \in A \vee x \in B$

{ Prop. of $\cup$ on (6): }

(7) $x \in A \cup B$

{ Property of $=$ on (7) and (1): }

(8) $x \in A$

{ $\neg$-elim on (8) and (5): }

(9) False

{ $\forall$-intro on (2) and (9): }

(10) $\forall_x[x \in B \setminus A : \texttt{False}]$

{ Prop. of $\emptyset$ on (10): }

(11) $B \setminus A = \emptyset$

{ $\Rightarrow$-intro on (1) and (11): }

(12) $A \cup B = A \Rightarrow B \setminus A = \emptyset$

The proof on the left is now complete, and it establishes that

$$A \cup B = A \Rightarrow B \setminus A = \emptyset$$

for all sets A and B.

We shall now incrementally rebuild the derivation, and simultaneously construct a proof in natural language:

To prove the implication $A \cup B = A \Rightarrow B \setminus A = \emptyset$, we assume that $A \cup B = A$ and establish that $B \setminus A = \emptyset$.

Note that, to establish $B \setminus A = \emptyset$, it suffices, by the Property of $\emptyset$, to consider an arbitrary element $x \in B \setminus A$ and derive a contradiction.

From $x \in B \setminus A$, it follows, by the Property of $\setminus$, that $x \in B$ and $\neg(x \in A)$.

From $x \in B$ it then follows, by the Property of $\cup$, that $x \in A \cup B$

Since $A \cup B = A$, it follows from $x \in A \cup B$, by the Property of $=$, that $x \in A$.

Note that we have now proved both $\neg(x \in A)$ and $x \in A$, so we have our contradiction. □