Solutions to selected exercises of Chapters 7–11

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September 16, 2015

This document contains solutions to the following exercises in the book [1]:

7.1, 7.2(c), 7.3(b), 7.7(b), 8.2(e), 8.7(c), 8.9(b),(d), 9.5(b), 9.6(b),11.4(c),11.6.

We **strongly** advise you to first try all these exercises by yourself, before looking at all at the solutions below. There is not a lot of variation possible in the way solutions to exercises should be written down. So if your solution in one way or another deviates from a solution below, then consider discussing the differences with your instructor.

7.1 [Comment: Due to the use of P and Q (which, in the book, stand for 'arbitrary abstract propositions') in parts (a), (c) and (d) of the exercise, and since a declaration or quantification for P and Q is lacking, it is not entirely clear what is meant. For parts (a) and (d) we show that the pairs of abstract propositions are comparable independent of which abstract propositions P and Q stand for. For parts (c), however, we consider two examples for P and Q: one for which the resulting abstract propositions are comparable, and one for which they are not.']

(a) Independent of for which abstract propositions P and Q stand, we have that $P \vee Q$ and $P \wedge Q$ are comparable. In fact, $P \wedge Q \stackrel{val}{\models} P \vee Q$ follows, e.g., from the following calculation:

$$
\begin{array}{ll}
 & P \wedge Q \\
\stackrel{val}{\models} & \{ \wedge\text{-}\vee\text{-weakening} \} \\
 & P \\
\stackrel{val}{\models} & \{ \wedge\text{-}\vee\text{-weakening} \} \\
 & P \vee Q \ .
\end{array}
$$

(b) The abstract propositions `False` and `True` are comparable: `False` $\stackrel{val}{\models}$ `True` follows, e.g., from the following calculation:

$$
\begin{array}{ll}
 & \texttt{False} \\
\stackrel{val}{\models} & \{ \text{Extremes} \} \\
 & \texttt{True} \ .
\end{array}
$$

(c) On the one hand, if $P = a$, with a a proposition variable, and $Q = b$, with b a proposition variable distinct from a, then P and $\neg(P \vee Q)$ are incomparable. To see that $P \stackrel{val}{\not\models} \neg(P \vee Q)$, note that if $a = 1$, then $P = 1$, but also $P \vee Q = 1$, so $\neg(P \vee Q) = 0$. To see that $\neg(P \vee Q) \stackrel{val}{\not\models} P$, note that if $a = 0$ and $b = 1$, then $P \vee Q = 0$, so $\neg(P \vee Q) = 1$, but $P = 0$.

On the other hand, if $P = \texttt{False}$, then, independent of what Q is, by Extremes it immediately follows that $P = \texttt{False} \stackrel{val}{\models} \neg(P \vee Q)$, so then P and $\neg(P \vee Q)$ are comparable.

(d) Independent of for which abstract propositions P and Q stand, we have that P and $\neg(P \Rightarrow Q)$ are comparable. In fact, $\neg(P \Rightarrow Q) \stackrel{val}{\models} P$ follows, e.g., from the following calculation:

$$
\begin{array}{ll}
 & \neg(P \Rightarrow Q) \\
\stackrel{val}{==} & \{ \text{ Implication } \} \\
 & \neg(\neg P \vee Q) \\
\stackrel{val}{==} & \{ \text{ De Morgan } \} \\
 & \neg\neg P \wedge \neg Q \\
\stackrel{val}{==} & \{ \text{ Double Negation } \} \\
 & P \wedge \neg Q \\
\stackrel{val}{\models} & \{ \wedge\text{-}\vee\text{-weakening } \} \\
 & P
\end{array}
$$

7.2 (c) The proposition is not valid for all abstract propositions P, Q and R. To see this, let a and b are distinct propositional variables and let $P = a \wedge b$, $Q = a$ and $R = b$. Then both $P \stackrel{val}{\models} Q$ and $P \stackrel{val}{\models} R$ hold (by $\wedge$-$\vee$-weakening), but $Q \stackrel{val}{\neq} R$.

7.3 (b) We show with a calculation that $\neg(P \Rightarrow \neg Q) \stackrel{val}{\models} (P \vee R) \wedge Q$:

$$
\begin{array}{ll}
 & \neg(P \Rightarrow \neg Q) \\
\stackrel{val}{==} & \{ \text{ Implication } \} \\
 & \neg(\neg P \vee \neg Q) \\
\stackrel{val}{==} & \{ \text{ De Morgan } \} \\
 & \neg\neg P \wedge \neg\neg Q \\
\stackrel{val}{==} & \{ \text{ Double Negation } (2\times) \} \\
 & P \wedge Q \\
\stackrel{val}{\models} & \{ \wedge\text{-}\vee\text{-weakening+Monotony } \} \\
 & (P \vee R) \wedge Q \ .
\end{array}
$$

7.7 (b) By Lemma 7.3.4, it suffices to establish that

$$P \Rightarrow ((\neg P \vee Q) \wedge (P \Leftrightarrow Q)) \stackrel{val}{\models} P \Leftrightarrow P \ .$$

This is achieved with the following calculation:

$$
\begin{array}{ll}
 & P \Rightarrow ((\neg P \vee Q) \wedge (P \Leftrightarrow Q)) \\
\stackrel{val}{\models} & \{ \text{ Extremes } \} \\
 & \texttt{True} \\
\stackrel{val}{==} & \{ \text{ Self-equivalence } \} \\
 & P \Leftrightarrow P \ .
\end{array}
$$

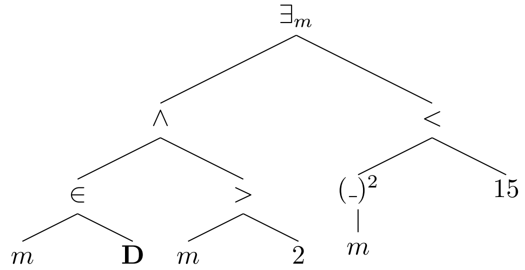
8.2 (e) $\exists_x[x \in \mathbf{M}:$
$\quad Younger(x, \text{Bernard}) \wedge$
$\quad Man(x) \wedge$
$\quad \forall_y[y \in \mathbf{M} : Child(x, y) \Leftrightarrow Child(\text{Bernard}, y)]]$

There is a person that is younger than Bernard and male and with the same parents.

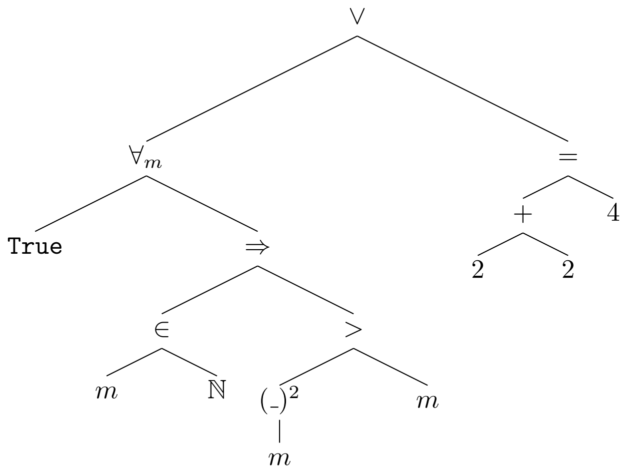
Comment: The formula above expresses the predicate "x is a sibling of y" as $\forall_z[z \in \mathbf{M} : Child(x, z) \Leftrightarrow Child(y, z)]$. This formalisation is based on the interpretation that x and y are siblings if they have the same parents, and (implicitly) assumes that every person has parents (for: people without parents are siblings according to this formula!). An alternative interpretation of "x is a sibling of y" could be "x and y have a parent in common", which can be formalised as $\exists_z[z \in \mathbf{M} : Child(x, z) \wedge Child(y, z)]$. Note that this formulation does imply that x and y are siblings if they only have one parent in common (i.e., they are actually 'half-siblings').

8.7 (c) $\exists_x[x \in \mathbf{D} : \forall_y[y \in \mathbf{D} : x = y]]$.

8.9 (b) The tree associated with $\exists_m[(m \in \mathbf{D}) \wedge (m > 2) : m^2 < 15]$ is:



(d) The tree associated with $\forall_m[m \in \mathbb{N} \Rightarrow m^2 > m] \vee (2 + 2 = 4)$ is



9.5 (b) The equivalence $\forall_k[P : Q \vee R] \stackrel{val}{==} \forall_k[P \wedge \neg Q : R]$ is established by the following calculation:

$$\begin{aligned}
& \forall_k[P \wedge \neg Q : R] \\
\overset{val}{=\!=} \; & \{ \text{ Domain Weakening } \} \\
& \forall_k[P : \neg Q \Rightarrow R] \\
\overset{val}{=\!=} \; & \{ \text{ Implication } \} \\
& \forall_k[P : \neg\neg Q \vee R] \\
\overset{val}{=\!=} \; & \{ \text{ Double Negation } \} \\
& \forall_k[P : Q \vee R]
\end{aligned}$$

9.6 (b) To show that $\exists_k[P : Q] \wedge \exists_k[P : R] \overset{val}{\neq} \exists_k[P : Q \wedge R]$ we need to find a counterexample, i.e., concrete predicates P, Q and R for which the equivalence does not hold.

Let $P = (k \in \mathbb{Z})$, let $Q = (k > 0)$ and let $R = (k < 0)$. Then, since $1 \in \mathbb{Z}$ and $1 > 0$, the proposition $\exists_k[P : Q]$ is true, and since $-1 \in \mathbb{Z}$ and $-1 < 0$, the proposition $\exists_k[P : R]$ is true. But $\exists_k[P : Q \wedge R]$ is not true, for there does not exist an integer that is both positive and negative.

11.4 (c) The proposition is true, for 29 is a prime number that is 1 plus a multiple of 7 ($29 = 1 + 4 \cdot 7$).

11.6 We need to prove that the square of an odd integer is 1 plus a multiple of 8.

To this end, let n be the square of an odd integer. Then there exists $x \in \mathbb{Z}$ such that $n = (2x + 1)^2 = 4x^2 + 4x + 1$. Clearly, it now remains to establish that $4x^2 + 4x$ is a multiple of eight; we distinguish two cases:

(a) If x is even, then there exists $y \in \mathbb{Z}$ such that $x = 2y$, so

$$4x^2 + 4x = 4(2y)^2 + 4 \cdot 2y = 16y^2 + 8y = 8(2y^2 + y) \ .$$

(b) If x is odd, then there exists $y \in \mathbb{Z}$ such that $x = 2y + 1$, so

$$\begin{aligned}
4x^2 + 4x = 4(2y + 1)^2 + 4(2y + 1) &= 4(4y^2 + 4y + 1) + 4(2y + 1) \\
&= 16y^2 + 24y + 8 = 8(2y^2 + 3y + 1) \ .
\end{aligned}$$

In both cases it is clear that $4x^2 + 4x$ is indeed a multiple of 8.

References

[1] Rob Nederpelt and Fairouz Kamareddine. *Logical Reasoning: A First Course*, volume 3 of *Texts in Computing*. King's College Publications, second revised edition edition, 2011.