Solutions to selected exercises of Chapters 12–15

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September 25, 2017

This document contains solutions to the following exercises in the book [1]:

12.4(d), 13.1(b), 13.2(b), 14.2(a), 14.5(a), 14.6(b), 14.8(b,c), 14.9(b), 14.10(b), 15.5, 15.8, 15.9.

We **strongly** advise you to first try all these exercises by yourself, before looking at all at the solutions below. There is not a lot of variation possible in the way solutions to exercises should be written down. So if your solution in one way or another deviates from a solution below, then consider discussing the differences with your instructor.

12.4 (d) The following derivation shows that the formula

$$(P \Rightarrow Q) \Rightarrow ((R \Rightarrow (P \Rightarrow Q)) \wedge ((P \wedge R) \Rightarrow Q))$$

is a tautology:

{ Assume; }

(1) $\boxed{P \Rightarrow Q}$

{ Assume: }

(2) $\boxed{R}$

{ Still valid: }

(3) $P \Rightarrow Q$

{ $\Rightarrow$-intro on (2) and (3): }

(4) $R \Rightarrow (P \Rightarrow Q)$

{ Assume: }

(5) $\boxed{P \wedge R}$

{ $\wedge$-elim on (5): }

(6) P

{ $\Rightarrow$-elim on (1) and (6): }

(7) Q

{ $\Rightarrow$-intro on (5) and (7): }

(8) $(P \wedge R) \Rightarrow Q$

{ $\wedge$-intro on (4) and (8): }

(9) $(R \Rightarrow (P \Rightarrow Q)) \wedge ((P \wedge R) \Rightarrow Q)$

{ $\Rightarrow$-intro on (1) and (9): }

(10) $(P \Rightarrow Q) \Rightarrow ((R \Rightarrow (P \Rightarrow Q)) \wedge ((P \wedge R) \Rightarrow Q))$

13.1 (b)
(1): valid from (1) to (9)
(2): valid from (2) to (3)
(3): only valid on (3)
(4): valid from (4) to (9)
(5): valid from (5) to (7)
(6): valid from (6) to (7)
(7): only valid on (7)
(8): valid from (8) to (9)
(9): only valid on (9)
(10): valid forever.

13.2 (b)
(1): context consists of hypothesis on (1)
(2): context consists of hypotheses on (1) and (2)
(3): context consists of hypotheses on (1) and (2)
(4): context consists of hypothesis on (1)
(5): context consists of hypotheses on (1) and (5)
(6): context consists of hypotheses on (1) and (5)
(7): context consists of hypotheses on (1) and (5)
(8): context consists of hypothesis on (1)
(9): context consists of hypothesis on (1)
(10): context is empty.

14.2 (a) The following derivation shows that the formula

$$(P \Rightarrow \neg Q) \Rightarrow ((P \Rightarrow Q) \Rightarrow \neg P)$$

is a tautology:

{ Assume: }

(1) $\boxed{P \Rightarrow \neg Q}$

{ Assume: }

(2) $\boxed{P \Rightarrow Q}$

{ Assume: }

(3) $\boxed{P}$

{ $\Rightarrow$-elim on (1) and (3): }

(4) $\neg Q$

{ $\Rightarrow$-elim on (2) and (3): }

(5) Q

{ $\neg$-elim on (4) and (5): }

(6) False

{ ¬-intro on (3) and (6): }

(7) $\neg P$

{ ⇒-intro on (2) and (7): }

(8) $(P \Rightarrow Q) \Rightarrow \neg P$

{ ⇒-intro on (1) and (8): }

(9) $(P \Rightarrow \neg Q) \Rightarrow ((P \Rightarrow Q) \Rightarrow \neg P)$

14.5 (a) The following derivation shows that the formula

$$(\neg P \Rightarrow P) \Rightarrow P$$

is a tautology.

{ Assume: }

(1) $\boxed{\neg P \Rightarrow P}$

{ Assume: }

(2) $\boxed{\neg P}$

{ ⇒-elim on (1) and (2): }

(3) P

{ ¬-elim on (2) and (3): }

(4) False

{ RAA[1]on (2) and (4): }

(5) P

{ ⇒-intro on (1) and (5): }

(6) $(\neg P \Rightarrow P) \Rightarrow P$

14.6 (b) The following derivation shows that the formula

$$(P \Rightarrow Q) \vee \neg Q$$

is a tautology:

{ Assume: }

(1) $\boxed{\neg(P \Rightarrow Q)}$

{ Assume: }

[1]RAA stands for 'Reductio Ad Absurdum' (proof by contradiction). In [1] the rule is referred to by "¬-intro followed by ¬¬-elim."

(2) $\boxed{Q}$

{ Assume: }

(3) $\boxed{P}$

{ Still valid: }

(4) Q

{ $\Rightarrow$-intro on (3) and (4): }

(5) $P \Rightarrow Q$

{ $\neg$-elim on (1) and (5): }

(6) `False`

{ $\neg$-intro on (2) and (6): }

(7) $\neg Q$

{ $\vee$-intro on (1) and (7): }

(8) $(P \Rightarrow Q) \vee \neg Q$

14.8 (b) We prove that $(x \geq 2 \vee x = -1) \Rightarrow x^3 - 3x - 2 \geq 0$ with the help of case distinction:

{ Assume: }

(1) $\boxed{x \geq 2 \vee x = -1}$

{ Assume: }

(2) $\boxed{x \geq 2}$

{ Mathematics on (2): }

(3) $(x - 2) \geq 0$

{ Mathematics on (2): }

(4) $(x + 1) \geq 0$

{ Mathematics on (3) and (4): }

(5) $x^3 - 3x - 2 = (x - 2)(x + 1)(x + 1) \geq 0$

{ $\Rightarrow$-intro on (2) and (5): }

(6) $(x \geq 2) \Rightarrow (x^3 - 3x + 2 \geq 0)$

{ Assume: }

(7) $\boxed{x = -1}$

{ Substitute -1 for x: }

(8) $x^3 - 3x - 2 = (-1)^3 - 3 \cdot (-1) - 2 = 0 \geq 0$

{ $\Rightarrow$-intro on (7) and (8): }

(9) $(x = -1) \Rightarrow (x^3 - 3x - 2 \geq 0)$

{ Case distinction on (1), (6) and (9): }

(10) $x^3 - 3x - 2 \geq 0$

{ $\Rightarrow$-intro on (1) and (11): }

(11) $(x \geq 2 \vee x = -1) \Rightarrow (x^3 - 3x - 2 \geq 0)$

The case-distinction tautology is used with $P = (x \geq 2)$, $Q = (x = -1)$ and $R = (x^3 - 3x - 2 \geq 0)$. (NB: the exercise does not explicitly ask for a derivation, so the argument may be written otherwise, but it should precisely indicate how case distinction is used; see Remark 14.8.1 in the book [1] for an example of a more informal argument.)

(c) We need to prove that $x^2 = y^2 \Leftrightarrow (x = y \vee x = -y)$, using case distinction.

To prove the bi-implication, we establish both the implication from left to right (i.e., $x^2 = y^2 \Rightarrow (x = y \vee x = -y)$) and the implication from right to left (i.e., $(x = y \vee x = -y) \Rightarrow x^2 = y^2$) separately.

To prove the implication from left to right we do not need case distinction. Suppose that $x^2 = y^2$. Then $x = \sqrt{y^2} = y$ or $x = -\sqrt{y^2} = -y$. So $x = y \vee x = -y$. Thereby, we have established the implication $(x^2 = y^2) \Rightarrow (x = y \vee x = -y)$.

To prove the implication from right to left, suppose that $x = y \vee x = -y$. We now use case distinction to establish $x^2 = y^2$, taking $P = (x = y)$, $Q = (x = -y)$ and $R = (x^2 = y^2)$. Note that $P \vee Q$ holds by the supposition that $x = y \vee x = -y$. To see, on the one hand, that $P \Rightarrow R$ holds, suppose that $x = y$. Then it immediately follows that $x^2 = y^2$ holds too. On the other hand, to see that $Q \Rightarrow R$ holds, suppose that $x = -y$. Then $x^2 = (-y)^2 = y^2$. We conclude that $P \vee Q$, $P \Rightarrow R$ and $Q \Rightarrow R$, so, by case distinction, R, i.e., $x^2 = y^2$, holds.

14.9 (b) The following derivation shows that the formula

$$(P \Leftrightarrow Q) \Rightarrow (\neg P \Leftrightarrow \neg Q)$$

is a tautology:

{ Assume: }

(1) $\boxed{P \Leftrightarrow Q}$

{ Assume: }

(2) $\boxed{\neg P}$

{ Assume: }

(3) $\boxed{Q}$

{ $\Leftrightarrow$-elim on (1): }

(4) $Q \Rightarrow P$

{ $\Rightarrow$-elim on (4) and (3): }

(5) P

(6) { $\neg$-elim on (2) and (5): }

`False`

{ $\neg$-intro on (3) and (6): }

(7) $\neg Q$

{ $\Rightarrow$-intro on (2) and (7): }

(8) $\neg P \Rightarrow \neg Q$

{ Assume: }

(9) $\boxed{\neg Q}$

{ Assume: }

(10) $\boxed{P}$

{ $\Leftrightarrow$-elim on (1): }

(11) $P \Rightarrow Q$

{ $\Rightarrow$-elim on (11) and (10): }

(12) Q

{ $\neg$-elim on (9) and (12): }

(13) `False`

{ $\neg$-intro on (10) and (13): }

(14) $\neg P$

{ $\Rightarrow$-intro on (9) and (14): }

(15) $\neg Q \Rightarrow \neg P$

{ $\Leftrightarrow$-intro on (8) and (15): }

(16) $\neg P \Leftrightarrow \neg Q$

{ $\Rightarrow$-intro on (1) and (16): }

(17) $(P \Leftrightarrow Q) \Rightarrow (\neg P \Leftrightarrow \neg Q)$

14.10 (b) To prove with a calculation that the formula $(P \Leftrightarrow Q) \Rightarrow (\neg P \Leftrightarrow \neg Q)$ is a tautology, it suffices to establish with a calculation that

$$P \Leftrightarrow Q \overset{val}{\models} \neg P \Leftrightarrow \neg Q \quad :$$

$(P \Leftrightarrow Q)$

$\stackrel{val}{=\!=}$ { Bi-implication }

$(P \Rightarrow Q) \wedge (Q \Rightarrow P)$

$\stackrel{val}{=\!=}$ { Contraposition (2×) }

$(\neg Q \Rightarrow \neg P) \wedge (\neg P \Rightarrow \neg Q)$

$\stackrel{val}{=\!=}$ { Bi-implication }

$\neg P \Leftrightarrow \neg Q$

(NB: the calculation above actually establishes the stronger result that $(P \Leftrightarrow Q) \stackrel{val}{=\!=} (\neg P \Leftrightarrow \neg Q)$ from which it follows that $(P \Leftrightarrow Q) \Leftrightarrow (\neg P \Leftrightarrow \neg Q)$ is a tautology.)

15.5 In the solutions below we do use the *alternative methods* for existential quantification as described in Section 15.4 (thereby ignoring the suggestion of the book to use the methods for existential quantification described in Section 15.3):

(a) We rename the x and y bound by the universal quantifiers respectively to u and v, resulting in the formula: $\neg\exists_x\exists_y[P(x, y)] \Rightarrow \forall_u\forall_v[\neg P(u, v)]$, and then establish with a derivation that it is a tautology:

{ Assume: }

(1) $\neg\exists_x\exists_y[P(x, y)]$

{ Assume: }

(2) **var** u; `True`

{ Assume: }

(3) **var** v; `True`

{ Assume: }

(4) $P(u, v)$

{ ∃*-into on (4): }

(5) $\exists_y[P(u, y)]$

{ ∃*-intro on (5): }

(6) $\exists_x\exists_y[P(x, y)]$

{ ¬-elim on (1) and (6): }

(7) `False`

{ ¬-intro on (4) and (7): }

(8) $\neg P(u, v)$

{ ∀-intro on (3) and (8): }

(9) $\forall_v[\neg P(u, v)]$

{ $\forall$-intro on (2) and (9): }

(10) $\forall_u \forall_v[\neg P(u,v)]$

{ $\Rightarrow$-intro on (1) and (10): }

(11) $\neg \exists_x \exists_y [P(x,y)] \Rightarrow \forall_u \forall_v[\neg P(u,v)]$

(b) We rename the occurrences of x bound by the existential quantfiers respectively to y and z, resulting in the formula:

$$\forall_x[P(x) : Q(x)] \Rightarrow (\exists_y[P(y)] \Rightarrow \exists_z[Q(z)]) \ ,$$

and then establish with a derivation that it is a tautology:

{ Assume: }

(1) $\boxed{\forall_x[P(x) : Q(x)]}$

{ Assume: }

(2) $\boxed{\exists_y[P(y)]}$

{ $\exists^*$-elim on (2): }

(3) Pick an y with $P(y)$

{ $\forall$-elim on (1) and (3): }

(4) $Q(y)$

{ $\exists^*$-intro on (4): }

(5) $\exists_z[Q(z)]$

{ $\Rightarrow$-intro on (2) and (5): }

(6) $\exists_y[P(y)] \Rightarrow \exists_z[Q(z)]$

{ $\Rightarrow$-intro on (1) and (6): }

(7) $\forall_x[P(x) : Q(x)] \Rightarrow (\exists_y[P(y)] \Rightarrow \exists_z[Q(z)])$

15.8 (a) We prove with a derivation that the formula $\exists_x \forall_y[P(x,y)] \Rightarrow \forall_v \exists_u[P(u,v)]$ is a tautology:

{ Assume: }

(1) $\boxed{\exists_x \forall_y[P(x,y)]}$

{ Assume: }

(2) **var** v; `True`

{ $\exists^*$-elim on (1): }

(3) Pick an x with (`True` and) $\forall_y[P(x,y)]$

{ $\forall$-elim on (3) and (2): }

(4) $P(x, v)$

{ $\exists^*$-intro on (4): }

(5) $\exists_u[P(u, v)]$

{ $\forall$-intro on (2) and (5): }

(6) $\forall_v[\exists_u[P(u, v)]]$

{ $\Rightarrow$-intro on (1) and (6): }

(7) $\exists_x \forall_y [P(x, y)] \Rightarrow \forall_v \exists_u [P(u, v)]$

(b) We prove with a derivation that the formula

$$\forall_y[Q(y) \Rightarrow (P(y) \Rightarrow \exists_x[P(x) \wedge Q(x)])]$$

is a tautology:

{ Assume: }

(1) **var** y; `True`

{ Assume: }

(2) $Q(y)$

{ Assume: }

(3) $P(y)$

{ $\wedge$-intro on (3) and (2): }

(4) $P(y) \wedge Q(y)$

{ $\exists^*$-intro on (4): }

(5) $\exists_x[P(x) \wedge Q(x)]$

{ $\Rightarrow$-intro on (3) and (5): }

(6) $P(y) \Rightarrow \exists_x[P(x) \wedge Q(x)]$

{ $\Rightarrow$-intro on (2) and (6): }

(7) $Q(y) \Rightarrow (P(y) \Rightarrow \exists_x[P(x) \wedge Q(x)])$

{ $\forall$-intro on (1) and (7): }

(8) $\forall_y[Q(y) \Rightarrow (P(y) \Rightarrow \exists_x[P(x) \wedge Q(x)])]$

15.9 We prove with a derivation that the formula

$$(\forall_x[x \in \mathbb{N} : P(x)] \Rightarrow \exists_y[y \in \mathbb{N} : Q(y)]) \Rightarrow \exists_z[z \in \mathbb{N} : P(z) \Rightarrow Q(z)]$$

is a tautology.

{ Assume: }

(1) $\forall_x[x \in \mathbb{N} : P(x)] \Rightarrow \exists_y[y \in \mathbb{N} : Q(y)]$

{ Assume: }

(2) $\neg\exists_x[x \in \mathbb{N} : \neg P(x)]$

{ Assume: }

(3) **var** $x; x \in \mathbb{N}$

{ Assume: }

(4) $\neg P(x)$

{ $\exists^*$-intro on (3) and (4): }

(5) $\exists_x[x \in \mathbb{N} : \neg P(x)]$

{ $\neg$-elim on (2) and (5): }

(6) `False`

{ $\neg$-intro on (4) and (6), followed by $\neg\neg$-elim: }

(7) $P(x)$

{ $\forall$-intro on (3) and (7): }

(8) $\forall_x[x \in \mathbb{N} : P(x)]$

{ $\vee$-intro on (2) and (9): }

(9) $\forall_x[x \in \mathbb{N} : P(x)] \vee \exists_x[x \in \mathbb{N} : \neg P(x)]$

{ Assume: }

(10) $\forall_x[x \in \mathbb{N} : P(x)]$

{ $\Rightarrow$-elim on (1) and (10): }

(11) $\exists_y[y \in \mathbb{N} : Q(y)]$

{ $\exists^*$-elim on (11): }

(12) Pick a y with $y \in \mathbb{N}$ and $Q(y)$

{ Assume: }

(13) $P(y)$

{ Still valid: }

(14) $Q(y)$

{ $\Rightarrow$-intro on (13) and (14): }

(15) $P(y) \Rightarrow Q(y)$

{ $\exists^*$-intro on (12) and (15): }

(16) $\exists_z[z \in \mathbb{N} : P(z) \Rightarrow Q(z)]$

{ $\Rightarrow$-intro on (10) and (16): }

(17) $\forall_x[x \in \mathbb{N} : P(x)] \Rightarrow \exists_z[z \in \mathbb{N} : P(z) \Rightarrow Q(z)]$

{ Assume: }

(18) $\exists_x[x \in \mathbb{N} : \neg P(x)]$

{ $\exists^*$-elim on (18): }

(19) Pick an x with $x \in \mathbb{N}$ and $\neg P(x)$

{ Assume: }

(20) $P(x)$

{ $\neg$-elim on (19) and (20): }

(21) `False`

{ `False`-elim on (21): }

(22) $Q(x)$

{ $\Rightarrow$-intro on (20) and (22): }

(23) $P(x) \Rightarrow Q(x)$

{ $\exists^*$-intro on (19) and (23): }

(24) $\exists_z[z \in \mathbb{N} : P(z) \Rightarrow Q(z)]$

{ $\Rightarrow$-intro on (18) and (24): }

(25) $\exists_x[x \in \mathbb{N} : \neg P(x)] \Rightarrow \exists_z[z \in \mathbb{N} : P(z) \Rightarrow Q(z)]$

{ Case distinction on (9), (17) and (25): }

(26) $\exists_z[z \in \mathbb{N} : P(z) \Rightarrow Q(z)]$

{ $\Rightarrow$-intro on (1) and (27): }

(27) $(\forall_x[x \in \mathbb{N} : P(x)] \Rightarrow \exists_y[y \in \mathbb{N} : Q(y)]) \Rightarrow \exists_z[z \in \mathbb{N} : P(z) \Rightarrow Q(z)]$

References

[1] Rob Nederpelt and Fairouz Kamareddine. *Logical Reasoning: A First Course*, volume 3 of *Texts in Computing*. King's College Publications, second revised edition edition, 2011.