Solutions to selected exercises of Chapter 16

Bas Luttik

October 6, 2015

This document contains solutions to the following exercises in the book [1]:

16.4(d), 16.7(c,d), 16.8, 16.12(c,d).

We **strongly** advise you to first try all these exercises by yourself, before looking at all at the solutions below. There is not a lot of variation possible in the way solutions to exercises should be written down. So if your solution in one way or another deviates from a solution below, then consider discussing the differences with your instructor.

16.4 (d) We prove that $(A \subseteq B \wedge B \subseteq C) \Rightarrow (A \cup B) \subseteq C$ for all sets A, B and C:

{ Assume: }

(1) $A \subseteq B \wedge B \subseteq C$

{ $\wedge$-elim on (1): }

(2) $A \subseteq B$

{ $\wedge$-elim on (1): }

(3) $B \subseteq C$

{ Assume: }

(4) **var** x; $x \in A \cup B$

{ Property of $\cup$ on (4): }

(5) $x \in A \vee x \in B$

{ Assume: }

(6) $x \in A$

{ Property of $\subseteq$ on (2) and (6): }

(7) $x \in B$

{ Property of $\subseteq$ on (3) and (7): }

(8) $x \in C$

{ $\Rightarrow$-intro on (6) and (8): }

(9) $x \in A \Rightarrow x \in C$

{ Assume: }

(10) $x \in B$

{ Property of $\subseteq$ on (3) and (10): }

(11) $x \in C$

{ $\Rightarrow$-intro on (10) and (11): }

(12) $x \in B \Rightarrow x \in C$

{ Case distinction on (5), (9) and (12): }

(13) $x \in C$

{ $\forall$-intro on (4) and (13): }

(14) $\forall_x[x \in A \cup B : x \in C]$

{ Definition of $\subseteq$ on (14): }

(15) $A \cup B \subseteq C$

{ $\Rightarrow$-intro on (1) and (15): }

(16) $A \subseteq B \wedge B \subseteq C \Rightarrow A \cup B \subseteq C$

16.7 (c) We prove that $A \setminus C \subseteq ((A \setminus B) \cup (B \setminus C))$ for all sets A, B and C:

{ Assume: }

(1) **var** x; $x \in A \setminus C$

{ Assume: }

(2) $\neg(x \in A \setminus B)$

{ Property of $\setminus$ on (2): }

(3) $\neg(x \in A \wedge \neg(x \in B))$

{ De Morgan on (3): }

(4) $\neg(x \in A) \vee \neg\neg(x \in B)$

{ Property of $\setminus$ on (1): }

(5) $x \in A \wedge \neg(x \in C)$

{ $\wedge$-elim on (5): }

(6) $x \in A$

{ $\neg\neg$-intro on (6): }

(7) $\neg\neg(x \in A)$

{ $\vee$-elim on (4) and (7): }

(8) $\neg\neg(x \in B)$

{ Double Negation on (8): }

(9) $x \in B$

{ $\wedge$-elim on (5): }

(10) $\neg(x \in C)$

{ $\wedge$-intro on (9) and (10): }

(11) $x \in B \wedge \neg(x \in C)$

{ Property of $\setminus$ on (11): }

(12) $x \in B \setminus C$

{ $\vee$-intro on (2) and (12): }

(13) $x \in A \setminus B \vee x \in B \setminus C$

{ Property of $\cup$ on (13): }

(14) $x \in ((A \setminus B) \cup (B \setminus C))$

{ $\forall$-intro on (1) and (14): }

(15) $\forall_x[x \in A \setminus C : x \in ((A \setminus B) \cup (B \setminus C))]$

{ Definition of $\subseteq$ on (15): }

(16) $A \setminus C \subseteq ((A \setminus B) \cup (B \setminus C))$

(d) It suffices to prove, for arbitrary x, that

$$x \in A \setminus (B \setminus A) \stackrel{val}{==} x \in A \;;$$

we proceed with a calculation:

$$\begin{aligned} & x \in A \setminus (B \setminus A) \\ \stackrel{val}{==} \; & \{ \text{ Property of } \setminus \} \\ & x \in A \wedge \neg(x \in B \setminus A) \\ \stackrel{val}{==} \; & \{ \text{ Property of } \setminus \} \\ & x \in A \wedge \neg(x \in B \wedge \neg(x \in A)) \\ \stackrel{val}{==} \; & \{ \text{ De Morgan + Double Negation } \} \\ & x \in A \wedge (\neg(x \in B) \vee x \in A) \\ \stackrel{val}{==} \; & \{ \text{ Absorption}^* \} \\ & x \in A \;. \end{aligned}$$

* "Absorption" refers to the valid equivalence $P \wedge (P \vee Q) \stackrel{val}{==} P$, which is proved in Exercise 6.3(b).

16.8 (a) We prove that $A \cup B = \emptyset \Rightarrow A \cap B = \emptyset$ for all sets A, B and C:

{ Assume: }

(1) $\boxed{A \cup B = \emptyset}$

  { Assume: }

(2) $\boxed{\textbf{var } x;\ x \in A \cap B}$

   { Property of $\cap$ on (2): }

(3) $x \in A \wedge x \in B$

   { $\wedge$-$\vee$-weakening ($2\times$) on (3): }

(4) $x \in A \vee x \in B$

   { Property of $\cup$ on (4): }

(5) $x \in A \cup B$

   { Property of $=$ on (1) and (5): }

(6) $x \in \emptyset$

   { Property of $\emptyset$ on (6): }

(7) $\texttt{False}$

  { $\forall$-intro on (2) and (7): }

(8) $\forall_x[x \in A \cap B : \texttt{False}]$

  { Property of $\emptyset$ on (8): }

(9) $A \cap B = \emptyset$

{ $\Rightarrow$-intro on (1) and (9): }

(10) $A \cup B = \emptyset \Rightarrow A \cap B = \emptyset$

(b) We prove that $A \setminus B = \emptyset \Rightarrow A \subseteq B$ for all sets A, B and C:

{ Assume: }

(1) $\boxed{A \setminus B = \emptyset}$

  { Assume: }

(2) $\boxed{\textbf{var } x;\ x \in A}$

   { Assume }

(3) $\boxed{\neg(x \in B)}$

    { $\wedge$-intro on (2) and (3): }

(4) $x \in A \wedge \neg(x \in B)$

    { Property of $\setminus$ on (4): }

(5) $x \in A \setminus B$

{ Property of $\emptyset$ on (1) and (5): }

(6) $x \in \emptyset$

{ Property of $\emptyset$ on (6): }

(7) `False`

{ ¬-intro on (3) and (7) followed by ¬¬-elim: }

(8) $x \in B$

{ ∀-intro on (2) and (8): }

(9) $\forall_x[x \in A : x \in B]$

{ Definition of ⊆: }

(10) $A \subseteq B$

{ ⇒-intro on (1) and (11): }

(11) $A \setminus B = \emptyset \Rightarrow A \subseteq B$

16.12 (c) The formula $A \times A \subseteq B \times C \Rightarrow A \subseteq B \cap C$ is true for all sets A, B and C, so we give a proof:

{ Assume: }

(1) $A \times A \subseteq B \times C$

{ Assume: }

(2) **var** x; $x \in A$

{ ∧-intro on (2) and (2): }

(3) $x \in A \wedge x \in A$

{ Property of × on (3): }

(4) $(x, x) \in A \times A$

{ Property of ⊆ on (1) and (4): }

(5) $(x, x) \in B \times C$

{ Property of × on (5): }

(6) $x \in B \wedge x \in C$

{ Property of ∩ on (6): }

(7) $x \in B \cap C$

{ ∀-intro on (2) and (7): }

(8) $\forall_x[x \in A : x \in B \cap C]$

{ Definition of ⊆: }

(9) $A \subseteq B \cap C$

{ $\Rightarrow$-intro on (1) and (9): }

(10) $A \times A \subseteq B \times C \Rightarrow A \subseteq B \cap C$

(d) The formula $A \times B = C \times D \Rightarrow A = C$ does *not* hold for all sets A, B, C and D; we give a counterexample:

Let $A = \{0\}$, $C = \{1\}$ and $B = D = \emptyset$. Then

$$A \times B = \{0\} \times \emptyset = \emptyset = \{1\} \times \emptyset = C \times D \ ,$$

but $A = \{0\} \neq \{1\} = C$.

References

[1] Rob Nederpelt and Fairouz Kamareddine. *Logical Reasoning: A First Course*, volume 3 of *Texts in Computing*. King's College Publications, second revised edition edition, 2011.