Solutions to selected exercises about induction

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This document contains solutions to exercises in the book [1] and exercises from the set of Additional Exercises about induction, available as `http://www.win.tue.nl/~luttik/Courses/LV/handouts/Exercises.pdf`. We use the prefix AE to indicate that an exercise is from the set of Additional Exercises. The document contains solutions to the following exercises:

AE2, AE5, 19.7, 19.10, AE7, AE10 and AE14.

We **strongly** advise you to first try all these exercises by yourself, before looking at all at the solutions below. There is not a lot of variation possible in the way solutions to exercises should be written down. So if your solution in one way or another deviates from a solution below, then consider discussing the differences with your instructor.

The proofs below are given in textual form and contain the *minimum* amount of detail both with respect to the logic and reasoning involved. A proof by induction always has one or more basis cases and one or more step cases. The basis case do not involve an application of the induction hypothesis; the step cases do. The induction hypothesis should always be clear and explicitly stated. Moreover, it should be indicated in the proof where the induction hypothesis is used. We shall abbreviate 'induction hypothesis' with IH.

AE2 We prove that $\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$ for all positive natural numbers by induction on $n \geq 1$:

(Basis) If $n = 1$, then $\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{1}{1(1+1)} = \frac{1}{2} = \frac{n}{n+1}$.

(Step) Let $n \geq 1$, and suppose that

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1} \quad . \tag{IH}$$

Then

$$
\begin{aligned}
\sum_{i=1}^{n+1} \frac{1}{i(i+1)} &= \sum_{i=1}^{n} \frac{1}{i(i+1)} + \frac{1}{(n+1)(n+2)} \\
&= \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} && \text{(by IH)} \\
&= \frac{n(n+2)}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)} \\
&= \frac{n(n+2)+1}{(n+1)(n+2)} \\
&= \frac{n^2+2n+1}{(n+1)(n+2)} \\
&= \frac{(n+1)(n+1)}{(n+1)(n+2)} \\
&= \frac{n+1}{n+2} \ .
\end{aligned}
$$

AE5 We prove that $\sum_{i=1}^{n} i^k \leq \frac{1}{2}n^k(n+1)$ for all $k \geq 1$ and for all $n \geq 0$.

Let $n \geq 0$; we proceed by induction on $k \geq 1$:

(Basis) If $k = 1$, then, using that $\sum_{i=1}^{n} i = \frac{1}{2}n(n+1)$ as suggested in the exercise, $\sum_{i=1}^{n} i^k = \sum_{i=1}^{n} i = \frac{1}{2}n(n+1) = \frac{1}{2}n^k(n+1)$.

(Step) Let $k \geq 1$, and suppose that

$$
\sum_{i=1}^{n} i^k \leq \tfrac{1}{2}n^k(n+1) \ . \qquad \text{(IH)}
$$

Then

$$
\begin{aligned}
\sum_{i=1}^{n} i^{k+1} &\leq n \cdot \left(\sum_{i=1}^{n} i^k \right) \\
&\leq n \cdot \left(\tfrac{1}{2}n^k(n+1) \right) \quad \text{(by IH)} \\
&= \tfrac{1}{2}n^{k+1}(n+1) \ .
\end{aligned}
$$

19.7 We compute a few values for $a_0, a_1, a_2, \ldots$:

a_0	a_1	a_2	a_3	a_4	$\cdots$
2	3	6	15	42	$\cdots$

Note that $a_1 - a_0 = 1 = 3^0$, $a_2 - a_1 = 3$, $a_3 - a_2 = 9$, $a_4 - a_3 = 27$, $\ldots$. So it appears that $a_{i+1} - a_i = 3^i$ for all $i \in \mathbb{N}$; we prove this conjecture with induction on i:

(Basis) If $i = 0$, then $a_{i+1} - a_i = a_1 - a_0 = 1 = 3^0$, so, in this case, indeed $a_{i+1} - a_i = 3^i$.

(Step) Let $i \in \mathbb{N}$, and suppose that $a_{i+1} - a_i = 3^i$ (IH). Then

$$
a_{i+2} - a_{i+1} = (3a_{i+1} - 3) - (3a_i - 3) = 3(a_{i+1} - a_i) \overset{\text{IH}}{=} 3 \cdot 3^i = 3^{i+1} \ .
$$

This completes the proof that $a_{i+1} - a_i = 3^i$.

On the basis of $a_{i+1} - a_i = 3^i$, we conjecture that $a_n = x \cdot 3^n + y$ for some $x, y \in \mathbb{Z}$. To find the appropriate value for x, note that

$$\begin{aligned} x + y &= x \cdot 3^0 + y = a_0 = 2 \ , \text{ and} \\ 3x + y &= x \cdot 3^1 + y = a_1 = 3 \ , \end{aligned}$$

so $2x = (3x + y) - (x + y) = 3 - 2 = 1$, and hence $x = \frac{1}{2}$. To find the appropriate value for y, we simply replace x by its value $\frac{1}{2}$ in $x + y = 2$: $y = 2 - \frac{1}{2} = 1\frac{1}{2}$.

We proceed to prove, with induction on n, that $a_n = \frac{1}{2} \cdot 3^n + 1\frac{1}{2}$:

(BASIS) If $n = 0$, then $a_n = a_0 = 2 = \frac{1}{2} \cdot 3^0 + 1\frac{1}{2} = \frac{1}{2} \cdot 3^n + 1\frac{1}{2}$.

(STEP) Let $n \geq 0$, and suppose that $a_n = \frac{1}{2} \cdot 3^n + 1\frac{1}{2}$ (IH). Then

$$\begin{aligned} a_{n+1} &= 3a_n - 3 \\ &\overset{\text{IH}}{=} 3(\tfrac{1}{2} \cdot 3^n + 1\tfrac{1}{2}) - 3 \\ &= \tfrac{1}{2} \cdot 3^{n+1} + 4\tfrac{1}{2} - 3 \\ &= \tfrac{1}{2} \cdot 3^{n+1} + 1\tfrac{1}{2} \ . \end{aligned}$$

19.10 (a) The set $\{a_n \mid n \in \mathbb{N}\}$ is inductively defined by

$$\begin{aligned} a_0 &:= 0 \\ a_1 &:= 4 \\ a_{i+2} &:= 4(a_{i+1} - a_i) \quad (i \in \mathbb{N}) \end{aligned}$$

We prove with strong induction on n that $a_n = n \cdot 2^{n+1}$ for all $n \in \mathbb{N}$. Let $n \in \mathbb{N}$, and suppose that

$$a_k = k \cdot 2^{k+1} \text{ for all } k \in \mathbb{N} \text{ with } k < n \ . \qquad \text{(IH)}$$

We distinguish three cases:

Case $n = 0$: Then $a_n = a_0 = 0 = 0 \cdot 2^1 = n \cdot 2^{n+1}$.

Case $n = 1$: Then $a_n = a_1 = 4 = 1 \cdot 2^2 = n \cdot 2^{n+1}$.

Case $n \geq 2$: Then, since $0 \leq n - 2, n - 1 < n$, by IH it follows that $a_{n-2} = (n - 2) \cdot 2^{n-1}$ and $a_{n-1} = (n - 1) \cdot 2^n$. Hence,

$$\begin{aligned} a_n &= 4(a_{n-1} - a_{n-2}) \\ &= 4((n - 1) \cdot 2^n - (n - 2) \cdot 2^{n-1}) \\ &= 2(n - 1) \cdot 2^{n+1} - (n - 2) \cdot 2^{n+1} \\ &= n \cdot 2^{n+1} \ . \end{aligned}$$

Clearly, if $n \in \mathbb{N}$, then either $n = 0$, $n = 1$ or $n \geq 2$, so one of the cases applies, and in each case we have established that $a_n = n \cdot 2^{n+1}$.

(b) The set $\{a_n \mid n \in \mathbb{N}\}$ is inductively defined by

$$\begin{aligned} a_0 &:= 1 \\ a_1 &:= 1 \\ a_{i+2} &:= 5a_{i+1} - 6a_i \quad (i \in \mathbb{N}) \end{aligned}$$

We prove with strong induction on n that $a_n = 2^{n+1} - 3^n$ for all $n \in \mathbb{N}$. Let $n \in \mathbb{N}$, and suppose that

$$a_k = 2^{k+1} - 3^k \text{ for all } k \in \mathbb{N} \text{ with } k < n \ . \qquad \text{(IH)}$$

We distinguish three cases:

Case $n = 0$: Then $a_n = a_0 = 1 = 2 - 1 = 2^1 - 3^0 = 2^{n+1} - 3^n$.

Case $n = 1$: Then $a_n = a_1 = 1 = 4 - 3 = 2^2 - 3^1 = 2^{n+1} - 3^n$.

Case $n \geq 2$: Then, since $0 \leq n-2, n-1 < n$, by IH it follows that $a_{n-2} = 2^{n-1} - 3^{n-2}$ and $a_{n-1} = 2^n - 3^{n-1}$. Hence,

$$\begin{aligned} a_n &= 5a_{n-1} - 6a_{n-2} \\ &= 5(2^n - 3^{n-1}) - 6(2^{n-1} - 3^{n-2}) \\ &= 5 \cdot 2^n - 5 \cdot 3^{n-1} - 6 \cdot 2^{n-1} + 6 \cdot 3^{n-2} \\ &= 5 \cdot 2^n - 5 \cdot 3^{n-1} - 3 \cdot 2^n + 2 \cdot 3^{n-1} \\ &= 2 \cdot 2^n - 3 \cdot 3^{n-1} \\ &= 2^{n+1} - 3^n \quad . \end{aligned}$$

Clearly, if $n \in \mathbb{N}$, then either $n = 0$, $n = 1$ or $n \geq 2$, so one of the cases applies, and in each case we have established that $a_n = 2^{n+1} - 3^n$.

(c) The sequence $a_0, a_1, a_2, \ldots$ is inductively defined by

$$\begin{aligned} a_0 &:= 1 \\ a_{i+1} &:= \tfrac{1}{i+1}(a_0 + a_1 + \cdots + a_i) \quad (i \in \mathbb{N}) \end{aligned}$$

We prove with strong induction on n that $a_n = 1$ for all $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$, and suppose that $a_k = 1$ for all $k \in \mathbb{N}$ with $k < n$ (IH). We distinguish two cases:

Case $n = 0$: Then $a_n = a_0 = 1$, immediately by the definition of a_0.

Case $n \geq 1$: Then

$$a_n = \tfrac{1}{n}(a_0 + a_1 + \cdots + a_{n-1}) \stackrel{\text{IH}}{=} \tfrac{1}{n}(n \cdot 1) = 1 \quad .$$

Clearly, if $n \in \mathbb{N}$, then either $n = 0$ or $n \geq 1$, so one of the cases applies, and in each case we have established that $a_n = 1$. (Why is the case distinction necessary here too?)

(d) The sequence $a_0, a_1, a_2, \ldots$ is inductively defined by

$$\begin{aligned} a_0 &:= 1 \\ a_{i+1} &:= a_0 + \cdots + a_i - i + 1 \quad (i \in \mathbb{N}) \end{aligned}$$

We prove with induction on $n \geq 1$ that $a_n = 2^{n-1} + 1$ for all $n \in \mathbb{N}$.

(Basis) If $n = 1$, then $a_n = a_1 = 1 - 0 + 1 = 2 = 2^0 + 1 = 2^{n-1} + 1$.

(Step) Let $n \geq 1$, and suppose that $a_n = 2^{n-1} + 1$ (IH). Then

$$\begin{aligned} a_{n+1} &= a_0 + \cdots + a_n - n + 1 \\ &= a_0 + \cdots + a_{n-1} + a_n - n + 1 \\ &= a_0 + \cdots + a_{n-1} - (n-1) + 1 + a_n - 1 \\ &= a_n + a_n - 1 \\ &\stackrel{\text{IH}}{=} (2^{n-1} + 1) + (2^{n-1} + 1) - 1 \\ &= 2^n + 1 \quad . \end{aligned}$$

AE7 We prove by induction on $n \geq 0$ that $n^5 - n$ is divisible by 5.

(Basis) If $n = 0$, then $n^5 - n = 0^5 - 0 = 0 = 0 \cdot 5$, so $n^5 - n$ is divisible by 5.

(STEP) Let $n \geq 0$, and suppose that $n^5 - n$ is divisible by 5 (IH). By the induction hypothesis (IH), there exists $k \in \mathbb{Z}$ such that $n^5 - n = k \cdot 5$. Hence,

$$\begin{aligned}
(n+1)^5 - (n+1) &= (n^5 + 5n^4 + 10n^3 + 10n^2 + 5n + 1) - (n+1) \\
&= n^5 + 5n^4 + 10n^3 + 10n^2 + 4n \\
&= (5n^4 + 10n^3 + 10n^2 + 5n) + (n^5 - n) \\
&= (n^4 + 2n^3 + 2n^2 + n) \cdot 5 + k \cdot 5 \\
&= (n^4 + 2n^3 + 2n^2 + n + k) \cdot 5 \ ,
\end{aligned}$$

so $(n+1)^5 - (n+1)$ is a multiple of 5.

AE10 We prove with strong induction on p that any integer postage p greater than 5 cents can be formed by using only 2-cent and 7-cent stamps.

Let p be an arbitrary integer postage greater than 5, and suppose that all integer postages p' greater than 5, but smaller than p, can be formed using only 2-cent and 7-cent stamps (IH). We now distinguish three cases:

Case $p = 6$: Then postage p can be formed using three 2-cent stamps.

Case $p = 7$: Then postage p can be formed using one 7-cent stamp.

Case $p \geq 8$: Then $p - 2$ is a postage greater than 5 and smaller than p, so by the induction hypothesis $p-2$ can be formed using only 2-cent and 7-cent stamps. Clearly, if we add a 2-cent stamp to the formation of $p-2$, then we obtain a formation of p using only 2-cent and 7-cent stamps.

If p is greater than 5 cent, then clearly one of the cases applies, and in each of the three cases we have established that p can be formed using only 2-cent and 7-cent stamps. Thus, we have established that every postage greater than 5 cents can be formed using only 2-cent and 7-cent stamps.

AE14 (NB: In this exercise we denote by $d_0 \ldots d_n$ the number consisting of $n+1$ digits, with d_0 the first digit and d_n the last digit. Note that this means that $d_0 \ldots d_n = d_0 \cdot 10^n + \cdots + d_n \cdot 10^0$.)

We first prove that for all $n \in \mathbb{N}$ and for all $d_0, \ldots, d_n \in \{0, \ldots, 9\}$ there exists $k \in \mathbb{N}$ such that

$$d_0 \ldots d_n = 3 \cdot k + d_0 + \cdots + d_n \ . \tag{1}$$

The proof is by induction on n:

(BASIS) If $n = 0$ and $d_0 \in \{0, \ldots, 9\}$, then $d_0 \ldots d_n = d_0 = 0 + d_0 = 3 \cdot 0 + d_0$.

(STEP) Let $n \in \mathbb{N}$ and let $d_0, \ldots, d_n \in \{0, \ldots, 9\}$. Suppose that Equation (1) holds for these particular n and $d_0, \ldots, d_n$ (IH); we establish that there exists $\ell \in \mathbb{N}$ such that $d_0 \ldots d_n d_{n+1} = 3 \cdot \ell + d_0 + \cdots + d_n + d_{n+1}$:

$$\begin{aligned}
& d_0 \ldots d_n d_{n+1} \\
&\quad = 10 \cdot (d_0 \ldots d_n) + d_{n+1} \\
&\quad = 10 \cdot (3 \cdot k + d_0 + \cdots + d_n) + d_{n+1} && \text{(by IH)} \\
&\quad = 30 \cdot k + 10 \cdot (d_0 + \cdots + d_n) + d_{n+1} \\
&\quad = 30 \cdot k + 9 \cdot (d_0 + \cdots + d_n) + d_0 + \cdots + d_n + d_{n+1} \\
&\quad = 3(10 \cdot k + 3 \cdot (d_0 + \cdots + d_n)) + d_0 + \cdots + d_n + d_{n+1} \ .
\end{aligned}$$

Taking $\ell = 10 \cdot k + 3 \cdot (d_0 + \cdots + d_n)$, we have established that

$$d_0 \ldots d_n d_{n+1} = 3 \cdot \ell + d_0 + \cdots + d_n + d_{n+1} \ .$$

Using Equation (1) we now prove that, for all $n \in \mathbb{N}$ and for all $d_0, \ldots, d_n \in \{0, \ldots, 9\}$, the number $d_0 \ldots d_n$ is divisible by 3 if, and only if, the sum of its digits $d_0 + \cdots + d_n$ is divisible by 3.

To this end, let $n \in \mathbb{N}$ and $d_0, \ldots, d_n \in \{0, \ldots, 9\}$.

On the one hand, if $d_0 + \cdots + d_n$ is divisible by 3, then there exists ℓ such that $d_0 + \cdots + d_n = 3 \cdot \ell$. Hence, by Equation (1),

$$d_0 \ldots d_n = 3 \cdot k + 3 \cdot \ell = 3 \cdot (k + \ell) \ ,$$

so $d_0 \ldots d_n$ is divisible by 3.

On the other hand, if $d_0 \ldots d_n$ is divisible by 3, then there exists ℓ such that $d_0 \ldots d_n = 3 \cdot \ell$. Hence, by Equation (1),

$$d_0 + \cdots + d_n = d_0 \ldots d_n - 3 \cdot k = 3 \cdot \ell - 3 \cdot k = 3 \cdot (\ell - k) \ ,$$

so $d_0 + \cdots + d_n$ is divisible by 3.

Thus, we have now established, for all n and for all $d_0, \ldots, d_n \in \{0, \ldots, 9\}$, that the number $d_0 \ldots d_n$ is divisible by 3 if, and only if, the sum of the digits $d_0 + \cdots + d_n$ is divisible by three.

References

[1] Rob Nederpelt and Fairouz Kamareddine. *Logical Reasoning: A First Course*, volume 3 of *Texts in Computing*. King's College Publications, second revised edition edition, 2011.