Solutions to selected exercises about ordered sets

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This document contains solutions to exercises in the book [1]. The document contains solutions to the following exercises:

20.4, 20.6, 20.7, 20.13, 20.16, 20.17(a,c)

We **strongly** advise you to first try all these exercises by yourself, before looking at all at the solutions below. There is not a lot of variation possible in the way solutions to exercises should be written down. So if your solution in one way or another deviates from a solution below, then consider discussing the differences with your instructor.

The proofs below are given in textual form and contain the *minimum* amount of detail both with respect to the logic and reasoning involved.

20.4 (a) Define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) = x - 1 \qquad \text{for all } x \in \mathbb{R} \ ,$$

and define $h : \mathbb{R} \to \mathbb{R}$ by

$$h(x) = x + 1 \qquad \text{for all } x \in \mathbb{R} \ .$$

Then, for all $x \in \mathbb{R}$, $g(x) = x-1 < x = i(x)$ and $i(x) = x < x+1 = h(x)$. Hence $g < i < h$.

(b) To prove that $\langle \Phi, < \rangle$ is an irreflexive ordering, it suffices to prove that $<$ on Φ is irreflexive and transitive. (We do not need to prove that $<$ on Φ is strictly antisymmetric because, according to Exercise 20.2, this already follows from irreflexivity and transitivity.)

To prove that $<$ on Φ is irreflexive, let $f : \mathbb{R} \to \mathbb{R}$ be an arbitrary mapping in Φ and suppose that $f < f$; we derive a contradiction. Note that from the assumption that $f < f$ it follows by the definition of $<$ on Φ that $f(x) < f(x)$ for all $x \in \mathbb{R}$. But this contradicts the irreflexivity of $<$ on $\mathbb{R}$. We conclude that $\neg(f < f)$ for all $f \in \Phi$, so $<$ on Φ is irreflexive.

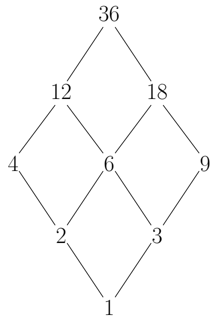
To prove that $<$ on Φ is transitive, let $f : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ be arbitrary mappings in Φ, and suppose that $f < g$ and $g < h$; we need to prove that $f < h$. To this end, let $x \in \mathbb{R}$. Then, since $f < g$, $f(x) < g(x)$, and, since $g < h$, $g(x) < h(x)$. Moreover, since $<$ on $\mathbb{R}$ is transitive, $f(x) < h(x)$. We have established that $f(x) < h(x)$ for all $x \in \mathbb{R}$, and hence $f < h$. We conclude that, for all $f, g, h \in \Phi$, if $f < g$ and $g < h$, then $f < h$, so $<$ on Φ is transitive.

(c) The ordering $<$ on Φ is *not* linear.

The ordering $<$ on Φ would be linear if, for every $f, g \in \Phi$, f and g are comparable, i.e., either $f < g$, $g < f$ or $f = g$. But this is not the case.

Counterexample: define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 2x$ for all $x \in \mathbb{R}$, and define $g : \mathbb{R} \to R$ by $g(x) = 3x$ for all $x \in \mathbb{R}$. Then $f(-1) = -2 > -3 = g(-1)$, so $\neg(f < g)$, and $g(1) = 3 > 2 = f(1)$, so $\neg(g < f)$, and both examples also show that $\neg(f = g)$. Hence, f and g are incomparable.

20.6 (a) The Hasse diagram for $\langle A_{36}, | \rangle$ is



(b) The Hasse diagram for $\langle A_n, | \rangle$ has the same structure as $\langle A_{36}, | \rangle$ if there are distinct prime numbers p and q such that $n = p^2 \cdot q^2$.

(c) The ordering $\langle A_n, | \rangle$ is linear if there exists a prime number p such that $n = p^k$ for some $k \in \mathbb{N}^+$.

20.7 (a) We need to prove that $\leq$ on $\{0,1\}^n$ is (i) reflexive, (ii) antisymmetric and (iii) transitive.

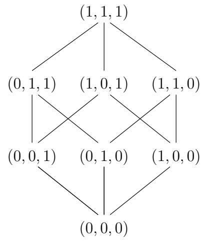
(i) Let $(x_1, \ldots, x_n)$ be an element of $\{0,1\}^n$. Then clearly $x_i \leq x_i$ for all $1 \leq i \leq n$, so $(x_1, \ldots, x_n) \leq (x_1, \ldots, x_n)$. Hence, $\leq$ is reflexive.

(ii) Let $(x_1, \ldots, x_n), (x'_1, \ldots, x'_n) \in \{0,1\}^n$. Suppose that $(x_1, \ldots, x_n) \leq (x'_1, \ldots, x'_n)$ and $(x'_1, \ldots, x'_n) \leq (x_1, \ldots, x_n)$. Then according to the definition of $\leq$ on $\{0,1\}^n$, for all $1 \leq i \leq n$, $x_i \leq x'_i$ and $x'_i \leq x_i$. Since $\leq$ on $\{0,1\}$ is antisymmetric, it follows that $x_i = x'_i$ for all $1 \leq i \leq n$, and hence $(x_1, \ldots, x_n) = (x'_1, \ldots, x'_n)$. We conclude that $\leq$ on $\{0,1\}^n$ is antisymmetric.

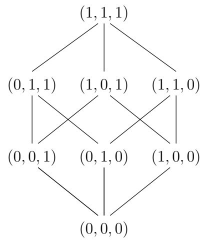
(iii) Let $(x_1, \ldots, x_n), (x'_1, \ldots, x'_n), (x''_1, \ldots, x''_n) \in \{0,1\}^n$, and suppose that $(x_1, \ldots, x_n) \leq (x'_1, \ldots, x'_n)$ and $(x'_1, \ldots, x'_n) \leq (x''_1, \ldots, x''_n)$. Then, according to the definition of $\leq$ on $\{0,1\}^n$, $x_i \leq x'_i$ and $x'_i \leq x''_i$ for all $1 \leq i \leq n$. Hence, since $\leq$ on $\{0,1\}$ is transitive, it follows that $x_i \leq x''_i$ for all $1 \leq i \leq n$. We conclude that $\leq$ on $\{0,1\}^n$ is transitive.

(b) The sequences $(\underbrace{0, \ldots, 0}_{n-1}, 1)$ and $(1, \underbrace{0, \ldots, 0}_{n-1})$ are incomparable.

(c) The Hasse diagram for $\langle \{0,1\}^3, \leq \rangle$ is



(d) The Hasse diagram for $\langle \mathcal{P}(\{1,2,3\}), \subseteq \rangle$ is



Note that the Hasse diagram for $\langle \mathcal{P}(\{1,2,3\}), \subseteq \rangle$ has exactly the same structure as the Hasse diagram for $\langle \{0,1\}^3, \leq \rangle$. This can be explained as follows: Let $f : \{0,1\}^3 \to \mathcal{P}(\{1,2,3\})$ be the mapping that associate with a sequence (x_1, x_2, x_3) the subset of $\{1,2,3\}$ that contains the number i $(1 \leq i \leq 3)$ if, and only if, $x_i = 1$. So, for instance, $f((0,0,0)) = \emptyset$, $f((1,0,0)) = \{1\}$, $f((0,1,0)) = \{2\}$, $f((0,0,1)) = \{3\}$, $f((1,1,0)) = \{1,2\}$, etc. Then f is a bijection, and $\leq$ on sequences corresponds with $\subseteq$ on subsets of $\{1,2,3\}$ via this bijection, in the sense that $(x_1, x_2, x_3) \leq (x_1', x_2', x_3')$ if, and only if, $f((x_1,, x_2, x_3)) \subseteq f((x_1', x_2', x_3'))$. (The mapping f is a bijection that also *preserves* the ordering; such a mapping is called an *order-isomorphism*.)

(e) For $n = p \cdot q \cdot r$ with p, q and r distinct prime numbers. (Can you define the order-isomorphism?)

20.13 (Below, we give a short proof in textual form; in the appendix there is a much more detailed proof in the form of a derivation. We encourage you to compare the proofs.)

Suppose that (x, y) is a maximal element of $A \times B$ in $\langle A \times B, \rangle, R_3\rangle$; we need to prove that then x is a maximal element of A in $\langle A, R_1 \rangle$ and y is a maximal element of B in $\langle B, R_2 \rangle$.

First, we prove that x is a maximal element of A in $\langle A, R_1 \rangle$. To this end, let $x' \in A$ and suppose that $x \; R_1 \; x'$; we establish that $x' = x$. Note that, since R_2 is reflexive, it holds that $y \; R_2 \; y$. So $x \; R_1 \; x'$ and $y \; R_2 \; y$, and hence, by the definition of R_3, it follows that $(x, y) \; R_3 \; (x', y)$. Since (x, y) is a maximal element of $A \times B$ in $\langle A \times B, R_3 \rangle$, we get $(x', y) = (x, y)$, and hence $x' = x$.

Then, it remains to prove that y is a maximal element of B in $\langle B, R_2 \rangle$. To this end, let $y' \in B$ and suppose that $y \; R_2 \; y'$; we establish that $y' = y$. Note that, since R_1 is reflexive, it holds that $x \; R_1 \; x$. So $x \; R_1 \; x$ and $y \; R_2 \; y'$, and hence, by the definition of R_3, it follows that $(x, y) \; R_3 \; (x', y)$. Since (x, y) is a maximal element of $A \times B$ in $\langle A \times B, R_3 \rangle$, we get $(x, y') = (x, y)$, and hence $y' = y$.

20.16 We consider $\langle \mathbb{N}^2, \leq_{\text{lex}} \rangle$, with $\leq_{\text{lex}}$ the lexicographic ordering associated with $\leq$ on $\mathbb{N}$.

(a) the greatest lower bound of $\{(x, y) \in \mathbb{N}^2 \mid x = 3\}$ in $\langle \mathbb{N}^2, \leq_{\text{lex}} \rangle$ is the pair $(3, 0)$; the least upper bound of $\{(x, y) \in \mathbb{N}^2 \mid x = 3\}$ in $\langle \mathbb{N}^2, \leq_{\text{lex}} \rangle$ is the pair $(4, 0)$.

(b) The greatest lower bound of $\{(x, y) \in \mathbb{N}^2 \mid y = 3\}$ in $\langle \mathbb{N}^2, \leq_{\text{lex}} \rangle$ is the pair $(0, 3)$; the least upper bound of $\{(x, y) \in \mathbb{N}^2 \mid y = 3\}$ in $\langle \mathbb{N}^2, \leq_{\text{lex}} \rangle$ does not exist.

(c) The greatest lower bound of $\{(x, y) \in \mathbb{N}^2 \mid x + y = 5\}$ in $\langle \mathbb{N}^2, \leq_{\text{lex}} \rangle$ is the pair $(0, 5)$; the least upper bound of $\{(x, y) \in \mathbb{N}^2 \mid x + y = 5\}$ in $\langle \mathbb{N}^2, \leq_{\text{lex}} \rangle$ is the pair $(5, 0)$.

20.17 (a) To prove that the irreflexive ordering S on $\mathbb{N}$ as defined in the exercise is linear, we need to prove that every two elements in $\mathbb{N}$ are comparable with respect to S. Let $k, \ell \in \mathbb{N}$; we distinguish three cases and prove that in each case either $k \; S \; \ell$, or $\ell \; S \; k$, or $k = \ell$:

(1) If k and ℓ are both even or k and ℓ are both odd, then note that, according to the definition of S, $k \; S \; \ell$ if, and only, $k < \ell$. It follows that either $k \; S \; \ell$ (if $k < \ell$) or $\ell \; S \; k$ (if $\ell < k$), or $k = \ell$.

(2) If k is even and ℓ odd, then according to the definition of S it holds that $k \; S \; \ell$.

(3) If k is odd and ℓ is even, then according to the definition of S it holds that $\ell \; S \; k$.

(c) Note, according to the ordering S, all even numbers precede all odd numbers, and both the even numbers and the odd numbers are ordered according to $<$. That is, if we list the elements of $\mathbb{N}$ in the order induced by S, we get:

$$0 \quad 2 \quad 4 \quad 6 \quad 8 \quad \cdots \quad 1 \quad 3 \quad 5 \quad 7 \quad 9 \quad \cdots$$

So every $n \in \mathbb{N}$ has a direct successor $n + 2$ according to S, and every $n \in \mathbb{N} \setminus \{0, 1\}$ has a direct predecessor $n - 2$. The numbers 0 and 1 do not have direct predecessors.

References

[1] Rob Nederpelt and Fairouz Kamareddine. *Logical Reasoning: A First Course*, volume 3 of *Texts in Computing*. King's College Publications, second revised edition edition, 2011.

A A logical derivation for 20.13

{ Assume: }

(1) $\boxed{\textbf{var } (x, y);\ (x, y) \in A \times B}$

{ Assume: }

(2) $\boxed{(x, y) \text{ is a maximal element of } A \times B}$

{ Property of $\times$ on (1): }

(3) $x \in A \wedge y \in B$

{ $\wedge$-elim on (3): }

(4) $x \in A$

{ $\wedge$-elim on (3): }

(5) $y \in B$

{ Def. maximal element on (2): }

(6) $\forall_{(x',y')}[(x', y') \in A \times B : (x, y)\ R_3\ (x', y') \Rightarrow (x', y') = (x, y)]$

{ Assume: }

(7) $\boxed{\textbf{var } x';\ x' \in A}$

{ Assume: }

(8) $\boxed{x\ R_1\ x'}$

{ $\wedge$-intro on (5) and (7): }

(9) $x' \in A \wedge y \in B$

{ Property of $\times$ on (9): }

(10) $(x', y) \in A \times B$

{ $\forall$-elim on (6) and (10): }

(11) $(x, y)\ R_3\ (x', y) \Rightarrow (x', y) = (x, y)$

{ Assume: }

(12) $\boxed{x = x'}$

{ R_2 is reflexive: }

(13) $y\ R_2\ y$

{ $\Rightarrow$-intro on (12) and (13): }

(14) $x = x' \Rightarrow y\ R_2\ y$

{ $\wedge$-intro on (8) and (14): }

(15) $x\ R_1\ x' \wedge (x = x' \Rightarrow y\ R_2\ y)$

{ Definition of R_3 on (15): }

(16) $(x, y)\ R_3\ (x', y)$

{ $\Rightarrow$-elim on (11) and (16): }

(17) $(x', y) = (x, y)$

{ Property of $\times$ on (17): }

(18) $x' = x \wedge y = y$

{ $\wedge$-elim on (18): }

(19) $x' = x$

{ $\Rightarrow$-intro on (8) and (19): }

(20) $x\ R_1\ x' \Rightarrow x' = x$

{ $\forall$-intro on (6) and (21): }

(21) $\forall_{x'}[x' \in A : x\ R_1\ x' \Rightarrow x' = x]$

{ Definition of maximal element: }

(22) x is a maximal element of A

{ Assume: }

(23) **var** y'; $y' \in B$

{ Assume: }

(24) $y\ R_2\ y'$

{ $\wedge$-intro on (4) and (23): }

(25) $x \in A \wedge y' \in B$

{ Property of $\times$ on (25): }

(26) $(x, y') \in A \times B$

{ $\forall$-elim on (6) and (26): }

(27) $(x, y)\ R_3\ (x', y) \Rightarrow (x', y) = (x, y)$

{ Assume: }

(28) $x = x$

{ (24) is still valid: }

(29) $y\ R_2\ y'$

{ $\Rightarrow$-intro on (24) and (29): }

(30) $x = x \Rightarrow y\ R_2\ y'$

{ $\wedge$-intro on (24) and (30): }

(31) $x\ R_1\ x \wedge (x = x \Rightarrow y\ R_2\ y')$

{ Definition of R_3 on (31): }

(32) $(x, y)\ R_3\ (x, y')$

{ $\Rightarrow$-elim on (27) and (32): }

(33) $(x, y') = (x, y)$

{ Property of $\times$ on (33): }

(34) $x = x \wedge y' = y$

{ $\wedge$-elim on (34): }

(35) $y' = y$

{ $\Rightarrow$-intro on (24) and (35): }

(36) $y\ R_2\ y' \Rightarrow y' = y$

{ $\forall$-intro on (23) and (36): }

(37) $\forall_{y'}[y' \in A : y\ R_2\ y' \Rightarrow y' = y]$

{ Definition of maximal element: }

(38) y is a maximal element of B

{ $\wedge$-intro on (22) and (38): }

(39) x is a maximal element of A and y is a maximal element of B