TECHNISCHE UNIVERSITEIT EINDHOVEN
Faculteit Wiskunde en Informatica

Final examination Logic & Set Theory (2IT61/2IT07)
(correction model)

Thursday October 30, 2014, 9:00–12:00 hrs.

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(2) 1. Prove that the formulas

$$a \Rightarrow (b \vee c) \quad \text{and} \quad (b \Rightarrow a) \wedge \neg c$$

are incomparable.

Solution: If $a = 0$ and $b = 1$, then, on the one hand, $a \Rightarrow (b \vee c) = 1$ and, on the other hand, $b \Rightarrow a = 0$ and hence $(b \Rightarrow a) \wedge \neg c = 0$. Thus we conclude that $a \Rightarrow (b \vee c) \overset{val}{\not\models} (b \Rightarrow a) \wedge \neg c$.

If $a = 1$ and $b = c = 0$, then, on the one hand, both $b \Rightarrow a = 1$ and $\neg c = 1$, so $(b \Rightarrow a) \wedge \neg c = 1$, and, on the other hand, $a = 1$ and $b \vee c = 0$, so $a \Rightarrow (b \vee c) = 0$. Thus we conclude that $(b \Rightarrow a) \wedge \neg c \overset{val}{\not\models} a \Rightarrow (b \vee c)$.

It follows that $a \Rightarrow (b \vee c)$ and $(b \Rightarrow a) \wedge \neg c$ are incomparable.

Correction suggestions: A correct answer (2 points) provides two assignments with arguments that one assignment shows that the left-hand side formula is not stronger than the right-hand side formula, and the other assignment shows that the right-hand side formula is not stronger than the left-hand side formula.

A complete truth table including both formulas with some extra annotations pointing out a row with a 1 in the column for the left-hand side formula and a 0 in the column for the right-hand side formula, and also vice versa, is worth 2 points.

An answer that only provides two correct assignments, but does not give any additional explanations, may be awarded with 1 point. A complete truth table without any additional explanations or indications is, hence, also worth 1 point.

An answer that only provides one of the two assignments (or a correct one and a wrong one), with decent explanations, is worth 1 point.

An answer that only provides one correct assignment and no further explanations is worth 0.5 point.

If the answer shows that the candidate knows the definition of incomparable, but fails to come up with correct assignments, then it should be awarded with 0.5 point.

(1) 2. Prove with a *calculation* (i.e., using the formal system based on standard equivalences described in *Part I* of the book) that

$$P \Rightarrow \neg Q \stackrel{val}{==} \neg(P \wedge Q) \ .$$

Solution: We have the following calculation:

$$\begin{array}{ll} & P \Rightarrow \neg Q \\ \stackrel{val}{==} & \{ \text{ Implication } \} \\ & \neg P \vee \neg Q \\ \stackrel{val}{==} & \{ \text{ De Morgan } \} \\ & \neg(P \wedge Q) \end{array}$$

Correction suggestions: A correct answer (1 point) provides a calculation based on standard equivalences.

If a correct calculation is given, but one or both names of standard equivalences are missing, then you may award 0.5 point.

If the candidate has forgotten to write the $\stackrel{val}{==}$ -symbol, but does list the formulas above in the correct order, mentioning the names of the applied standard equivalences in between, you may also award 0.5 point.

Omitting the curly brackets, or somehow misspelling the names of the standard equivalences, is not a big deal and should not lead to subtraction of points.

3. Let $\mathbb{P}$ be the set of all people, let Anna denote a particular person in $\mathbb{P}$, and let $\mathbb{B}$ be the set of all books. Furthermore, let R and L be predicates on $\mathbb{P} \times \mathbb{B}$ with the following interpretations for all $p \in \mathbb{P}$ and $b \in \mathbb{B}$:

$R(p, b)$ means 'p has read b', and
$L(p, b)$ means 'p liked b'.

Give formulas of predicate logic that express the following statements:

(1) (a) Everybody has read a book.

(1) (b) Anna has only read books she liked.

Solution:

(a) $\forall_p[p \in \mathbb{P} : \exists_b[b \in \mathbb{B} : R(p, b)]]$.

(b) $\forall_b[b \in \mathbb{B} \wedge R(\text{Anna}, b) : L(\text{Anna}, b)]$.

Correction suggestions:

(a) A correct answer (1 point) gives the formula above. Reasonable attempts that are not completely correct (e.g., $\forall_p[p \in \mathbb{P} : R(p, b)]$?), could perhaps still be awarded with 0.5 point. (To be awarded with more than 0 points, the formula should at least be syntactically correct.)

(b) A correct answer (1 point) gives the formula above. In this case, there are, perhaps, variations that also deserve 1 point, such as $\forall_b[b \in \mathbb{B} : R(\text{Anna}, b) \Rightarrow L(\text{Anna}, b)]$ and $\forall_b[b \in \mathbb{B} : R(\text{Anna}, b) \Leftrightarrow L(\text{Anna}, b)]$. Other reasonable, but incorrect, attempts could perhaps still be awarded with 0.5 point. Again, to be awarded with more than 0 points, the formula should at least be syntactically correct.

(3) 4. Prove with a *derivation* (i.e., using the methods described in *Part II* of the book) that the formula

$$(\forall_x[P(x) \Rightarrow \neg Q(x)] \wedge \exists_y[P(y) : Q(y) \vee R(y)]) \Rightarrow \exists_z[P(z) \wedge R(z)]$$

is a tautology.

Solution: We have the following derivation:

{ Assume: }

(1) $\forall_x[P(x) \Rightarrow \neg Q(x)] \wedge \exists_y[P(y) : Q(y) \vee R(y)]$

{ $\wedge$-elim on (1): }

(2) $\forall_x[P(x) \Rightarrow \neg Q(x)]$

{ $\wedge$-elim on (2): }

(3) $\exists_y[P(y) : Q(y) \vee R(y)]$

{ $\exists^*$-elim on (3): }

(4) Pick a y with $P(y)$ and $Q(y) \vee R(y)$

{ $\forall$-elim on (4) and (2): }

(5) $P(y) \Rightarrow \neg Q(y)$

{ $\Rightarrow$-elim on (5) and (4): }

(6) $\neg Q(y)$

{ $\vee$-elim on (4) and (6): }

(7) $R(y)$

{ $\wedge$-intro on (4) and (7): }

(8) $P(y) \wedge R(y)$

{ $\exists^*$-intro on (8): }

(9) $\exists_z[P(z) \wedge R(z)]$

{ $\Rightarrow$-intro on (1) and (9): }

(10) $(\forall_x[P(x) \Rightarrow \neg Q(x)] \wedge \exists_y[P(y) : Q(y) \vee R(y)]) \Rightarrow \exists_z[P(z) \wedge R(z)]$

Correction suggestions: A correct answer (3 points) gives a derivation showing that the formula is a tautology.

Let's not be too strict about omission of hints: if some or all hints are missing, subtract (at most) 0.5 points.

We should be stricter about wrong applications of rules. Subtract 1 point for every incorrect application of a rule (e.g., $\forall$-elim on the universal quantification without an appropriate object). An exception to this is when students forget to first to $\wedge$-elim before the $\forall$-elim or $\exists^*$-elim: I propose to subtract at most 0.5 for such misdemeanors.

Here is a tentative awarding of points to partly correct solutions:

- for correct $\Rightarrow$-intro: 0.5 point;
- for correct $\wedge$-elims and $\wedge$-intro: 0.5 point;
- for correct $\forall$-elim: 0.5 point;
- for correct $\exists^*$-elim and $\exists^*$-intro: 0.5 point;
- for correct $\Rightarrow$-elim: 0.5 point;
- for correct $\vee$-elim: 0.5 point.

(I'm not sure the scheme makes sense. It's just a suggestion, and you can alter it at will. As a rule of thumb, I think it is worse to apply a rule incorrectly, than to not apply the rule at all.)

(2) 5. Prove that the following formula holds for all sets A, B and C:

$$A \cap B = A \cap C \;\Rightarrow\; A \cap (B \setminus C) = \emptyset \ .$$

Solution: Let A, B and C be sets, and suppose that $A \cap B = A \cap C$. To prove that $A \cap (B \setminus C) = \emptyset$, it suffices, by the Property of $\emptyset$, to derive a contradiction from the assumption that $A \cap (B \setminus C)$ is non-empty. So let $x \in A \cap (B \setminus C)$. It then follows by the Properties of $\cap$ and $\setminus$ that $x \in A$, $x \in B$ and $\neg(x \in C)$. Hence, $x \in A \cap B$, so from the assumption $A \cap B = A \cap C$ it follows using Leibniz for set-theoretic equality that $x \in A \cap C$. But then, by the Property of $\cap$, we have that $x \in C$ and also $\neg(x \in C)$: a contradiction.

Correction suggestions: A correct answer (2 points) presents a proof of the property based on the properties and definitions pertaining to sets discussed in the course. There is no special requirement on how the proof is presented. In particular, we should also accept it in the form of a derivation. The proof need not explicitly refer to the rules of valid logical reasoning, but the underlying logical reasoning should, of course, be clear nevertheless.

I propose to award 1 point for the logical reasoning part of the proof (i.e., correctly dealing with implications, universal quantifications, negation, etc.), and 1 point for knowing and being able to apply the relevant basic definitions and properties pertaining to sets ($\emptyset$, $\cap$, $\setminus$).

Students have been told that they should name the basic definitions and properties pertaining to sets explicitly in proofs of properties of sets, but if they don't, then we should not be too strict, as long as it is clear which definitions and properties are applied where. Depending on whether just some or all names of the applied properties and definitions are missing, you may subtract between 0 and 0.5 points.

(3) 6. Let the sequence $a_0, a_1, a_2, \ldots$ be inductively defined by

$$\begin{aligned} a_0 &:= 0 \\ a_1 &:= 1 \\ a_{i+2} &:= 3a_{i+1} - 2a_i \qquad (i \in \mathbb{N}). \end{aligned}$$

Prove that $a_n = 2^n - 1$ for all $n \in \mathbb{N}$.

Solution: We prove that $a_n = 2^n - 1$ for all $n \in \mathbb{N}$ by strong induction on n. Let $n \in \mathbb{N}$, and suppose that for all $i \in \mathbb{N}$ such that $i < n$ it holds that $a_i = 2^i - 1$ (the induction hypothesis); we need to establish that $a_n = 2^n - 1$. To this end, we distinguish three cases:

- If $n = 0$, then $a_n = 0$ according to the definition of a_n, and since $2^n - 1 = 1 - 1 = 0$, it follows that $a_n = 2^n - 1$.
- If $n = 1$, then $a_n = 1$ according to the definition of a_n, and since $2^n - 1 = 2 - 1 = 1$, it follows that $a_n = 2^n - 1$.
- If $n \geq 2$, then $a_n = 3a_{n-1} - 2a_n$, according to the definition of a_n. Since $n-1, n-2 \in \mathbb{N}$ and $n-1, n-2 < n$, by the induction hypothesis $a_{n-1} = 2^{n-1} - 1$ and $a_{n-2} = 2^{n-2} - 1$, so

$$\begin{aligned} a_n = 3(2^{n-1} - 1) - 2(2^{n-2} - 1) &= 3(2^{n-1} - 1) - 2^{n-1} + 2 \\ &= 2 \cdot 2^{n-1} - 1 = 2^n - 1 \ . \end{aligned}$$

Correction suggestions: A correct answer (3 points) provides a proof that, given the inductive definition of the sequence $a_0, a_1, a_2, \ldots$, it holds that $a_n = 2^n - 1$ for all $n \in \mathbb{N}$. The proof may be presented in the form of a derivation.

Naturally, the proof is by strong induction, and candidates who are able to set up a proof by strong induction (that is, manage to reproduce the formula for strong induction and then are able to deal with the ensued logical reasoning, with a $\forall$-intro and an $\Rightarrow$-intro), earn 1 points by doing so. The remaining 2 points are then earned as follows:

- correctly dealing with the definition of the a_i, and, in particular, realising that there should be a case distinction, is worth 0.5 point;
- correct mathematical reasonings in the two basis cases is worth 0.5 point;
- correctly and explicitly using the induction hypothesis in the step case is worth 0.5 point;
- correctly doing the rest of the mathematical computation in the step case is worth 0.5 point.

For proofs that only deal with the first few cases explicitly, and then suggest that the property probably also holds in general, you may award (at most) 0.5 point (for correctly dealing with the basis cases).

7. Let A and B be sets, and let $F : A \to B$ and $G : B \to A$ be mappings such that $\forall_x[x \in A : G(F(x)) = x]$.

(2) (a) Prove that F is an injection.

(1) (b) Show, with a counterexample, that F is not necessarily a surjection.

<u>*Solution:*</u>

(a) To prove that F is an injection, let $x_1, x_2 \in A$ and suppose that $F(x_1) = F(x_2)$. Then, since $\forall_x[x \in A : G(F(x)) = x]$, we have that $G(F(x_1)) = x_1$ and $G(F(x_2)) = x_2$, so

$$x_1 = G(F(x_1)) = G(F(x_2)) = x_2 \ .$$

(b) Let $A = \mathbb{N}$ and $B = \mathbb{Z}$, let $F : A \to B$ be defined by $F(x) = x$ and $G : B \to A$ be defined by $G(x) = |x|$. Then, for all $x \in \mathbb{N}$, we have that $G(F(x)) = G(x) = |x| = x$. It is also clear, however, that F is not a surjection, for there, e.g., does not exist $x \in \mathbb{N}$ such that $F(x) = -1$.

<u>*Correction suggestions:*</u>

(a) A correct answer (2 points) provides a proof that F is an injection using the assumption that $\forall_x[x \in A : G(F(x)) = x]$. The proof may be presented in natural language, or as a derivation; in both cases, the logical reasoning should again be clear.

If the candidate gives the formula for '$F : A \to B$ is an injection' and nothing more, award 0.5 point.

If the candidate demonstrates that he or she knows when F is an injection by correctly setting up the reasoning by declaring two variables x_1 and x_2, assuming that $F(x_1) = F(x_2)$ and mentioning that the goal is to prove $x_1 = x_2$, then award 1 point.

Correct use of the property $\forall_x[x \in A : G(F(x)) = x]$ to conclude $x_1 = x_2$ is then also worth 1 point. For suboptimal arguments to this effect (e.g., from which it is not clear that the property $\forall_x[x \in A : G(F(x)) = x]$ is used), you could still award 0.5 point.

(So, a deriviation that proves the formula for F is an injection, except that it concludes from $F(x_1) = F(x_2)$ directly that $x_1 = x_2$, without explicitly using G, should be awarded with 1.5 points.)

(b) A correct answer (1 point) defines concrete sets A and B and mappings $F : A \to B$ and $G : B \to A$, argues that F and G satisfy the property $\forall_x[x \in A : G(F(x)) = x]$, and argues F is not a surjection.

For answers that correctly address some aspects but are unsatisfactory in some others (e.g., only correct concrete definitions of A, B, F and G, but no arguments, or only concrete A, B, and F with argument that F is not a surjection, but missing G and argument that F and G satisfy property $\forall_x[x \in A : G(F(x)) = x]$), you can still award 0.5 point.

8. We define

$$V := \mathcal{P}(\{0,1,2\}) \backslash \{\{0,1\},\{1,2\},\{0,1,2\}\} \ .$$

(1) (a) Determine V.

(2) (b) Make a Hasse diagram of $\langle V, \subseteq \rangle$.

(1) (c) What are the minimal elements of V in $\langle V, \subseteq \rangle$?
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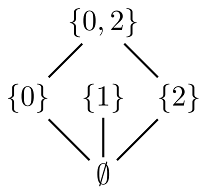
<u>*Solution:*</u>

(a) $V = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0,2\}\}$.

(b)



(c) The maximal elements of $\langle V, \subseteq \rangle$ are $\{1\}$ and $\{0,2\}$, and the minimal element of $\langle V, \subseteq \rangle$ is $\emptyset$.

<u>*Correction suggestions:*</u>

(a) A correct answer (1 point) gives V exactly as above.

If the answer somehow shows that the candidate knows how to compute $\mathcal{P}$, but there is, nevertheless, a mistake in the final answer, then you might still award 0.5 point.Bin

(b) A correct answer (2 points) gives the Hasse diagram. The Hasse diagram should, of course, not have fewer, or more connections between the elements. The relative positioning of connected elements (as above) should correctly indicate the 'direction' of the relation. Alternatively, the direction may also be indicated by an arrow instead of a connection without arrowheads, and then the relative positioning is not important.

Subtract 0.5 point for every missing or superfluous connection.

Subtract 0.5 point if the curly brackets are missing (or replace by parentheses).

(c) A correct answer (1 point) mentions that $\{1\}$ and $\{0,2\}$ are the maximal elements and $\emptyset$ is the minimal element.

If the specification of maximal and minimal elements is wrong but consistent with the Hasse diagram given by the candidate, you may 1 point.

If only the maximal elements are correctly specified, or only the minimal element is correctly specified, award 0.5 point.