TECHNISCHE UNIVERSITEIT EINDHOVEN
Faculteit Wiskunde en Informatica

Final exam Logic & Set Theory (2IT61)
(correction model)

Thursday November 4, 2016, 9:00–12:00 hrs.

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(2) 1. Determine whether the abstract propositions

$$\neg a \Leftrightarrow (a \vee b) \quad \text{and} \quad \neg a \wedge b$$

are *comparable* (i.e., the abstract proposition on the left is stronger than the abstract proposition on the right, or vice versa). Motivate your answer with a proof or a counterexample.

Solution: Consider an assignment that makes $\neg a \wedge b$ evaluate to true. Then this assignment should make both $\neg a$ and b evaluate to true, and hence the assignment must assign false to a and true to b. It then follows that, under this assignment, both $\neg a$ and $a \vee b$ evaluate to true, and hence $\neg a \Leftrightarrow (a \vee b)$ evaluates to true. Thus, we have established that every assignment that makes $\neg a \wedge b$ evaluate to true also makes $\neg a \Leftrightarrow (a \vee b)$ evaluate to true. It follows that $\neg a \Leftrightarrow (a \vee b)$ is a logical consequence of $\neg a \wedge b$, and therefore $\neg a \Leftrightarrow (a \vee b)$ and $\neg a \wedge b$ are comparable.

Correction suggestions: A correct answer (2 points) argues that the left-hand side abstract proposition is a logical consequence of the right-hand side abstract proposition, that the right-hand side abstract proposition is a logical consequence of the left-hand side abstract proposition, or that the abstract propositions are logically equivalent. The argument may reason about the shape of the abstract propositions (indirectly referring to the truth-table semantics of the connectives). Or it may consist a full truth table including columns for both abstract propositions with an explicit conclusion that the two abstract propositions are equivalent because their columns in the truth table are identical. Or it may consist of a calculation showing that the abstract propositions are equivalent or that one of the abstract propositions is a logical consequence of the other.

A mix between the two types of arguments (first simplifying the abstraction propositions by means of a calculation, and then drawing conclusions based truth-table semantics) is also fine.

Award 0.5 point if it is somehow clear from the answer that the candidate realises that the abstract propositions are indeed comparable and knows what this means. (The observation may, of course, be implicit in an argument showing equivalence or logical consequence, but if there is no decent argument then still 0.5 point may be awarded simply for the observation that the abstract propositions are comparable, because either abstract proposition is a logical consequence of the other.)

The remaining 1.5 point is awarded for the actual proof of comparability, as follows:

- If an argument referring to truth-table semantics is given, then it should (at least) make clear that whenever an assignment makes one of the two abstract propositions evaluate to true, then it also makes the other evaluate to true. Moreover, it should be clear from the argument (e.g., with a suitable conclusion on the basis of the truth table) *why* the truth table (or an argument implicitly referring to the truth table) demonstrates comparability. A correct argument could be: "from the truth table one can see that the abstract propositions are equivalent, and hence they are comparable". For a correct truth table alone, award 1 point; for a conclusion that shows that the candidate understands why the truth table demonstrates comparability, award 0.5 point.

So, if the candidate correctly concludes, on the basis of a correct truth table, that the abstract propositions are equivalent, but then concludes that the two abstract propositions are hence *incomparable* (i.e., the candidate does not understand that equivalent abstract propositions are comparable), then award 1 point in total.

- If a calculation is provided to demonstrate equivalence or logical consequence, then let's not punish the candidate for not mentioning explicitly the names of the standard equivalences they apply. There should, however, not be any confusion about which standard equivalence is applied where. Award 1.5 point for a correct calculation. Award 1 point for an argument that can be turned into a correct calculation by adding $\stackrel{val}{==}$ symbols here and there, or for a calculation with one wrong step. Award 0.5 point if the calculation is wrong, but still shows that the candidate has some understanding of what is required.

(2) 2. Give a formula of predicate logic that expresses the following sentence:

Every even natural number greater than 4 is the sum of two prime numbers.

(You may use $\mathbb{P}$ to denote the set of all prime numbers.)

Solution: The following formula of predicate logic expresses "Every even natural number greater than 4 is the sum of two prime numbers":

$$\forall_n[n \in \mathbb{N} \wedge \exists_m[m \in \mathbb{N} : n = 2 \cdot m] \wedge n > 4 : \exists_{p,q}[p \in \mathbb{P} \wedge q \in \mathbb{P} : n = p + q]] \ .$$

Correction suggestions: A correct answer, to be awarded with 2 points, gives the formula above or any formula that is *logically* equivalent to it. The predicates the candidate could use are not specified, so some flexibility in this respect is in order (e.g., 'n is even' may also be expressed as $n \mod 2 = 0$, or as $even(n)$). Note that, e.g., the formula

$$\forall_{m,n}[m \in \mathbb{N} \wedge n \in \mathbb{N} \wedge n = 2 \cdot m \wedge n > 4 : \exists_{p,q}[p \in \mathbb{P} \wedge q \in \mathbb{P} : n = p + q]]$$

is logically equivalent to the solution above, and should, hence, be awarded with 2 points.

Award 1 point for decent specification of the domain of the universal quantification (n is a natural number, n is even, $n > 4$). If some of the 'domain restrictions' are forgotten, subtract 0.5 point per omission. Note that, in view of logical equivalence, it is not necessary to put these restrictions in the domain part; they could also appear at the right of the implication.)

Award 1 point for correctly expressing the predicate 'n is the sum of two prime numbers'.

An answer that shows that the candidate does not understand the syntax of predicate logic in an essential way should not get any points. For minor syntactic issues (e.g., a missing bracket) you could just subtract 0.5 point.

(3) 3. Prove with a *derivation* (i.e., using the methods described in *Part II* of the book) that the formula

$$\exists_x[x \in \mathbb{N} : P(x) \vee Q(x)] \Rightarrow (\forall_y[y \in \mathbb{N} : \neg P(y)] \Rightarrow \exists_z[z \in \mathbb{N} : \neg Q(z) \Rightarrow R(z)])$$

is a tautology.

Solution: We have the following derivation:

{ Assume: }

(1) $\exists_x[x \in \mathbb{N} : P(x) \vee Q(x)]$

{ Assume: }

(2) $\forall_y[y \in \mathbb{N} : \neg P(y)]$

{ $\exists^*$-elim on (1): }

(3) Pick an x with $x \in \mathbb{N}$ and $P(x) \vee Q(x)$

{ Assume: }

(4) $\neg Q(x)$

{ $\forall$-elim on (2) and (3): }

(5) $\neg P(x)$

{ $\vee$-elim on (3) and (5): }

(6) $Q(x)$

{ $\neg$-elim on (6) and (4): }

(7) `False`

{ `False`-elim on (7): }

(8) $R(x)$

{ $\Rightarrow$-intro on (4) and (8): }

(9) $\neg Q(x) \Rightarrow R(x)$

{ $\exists^*$-intro on (3) and (9): }

(10) $\exists_z[z \in \mathbb{N} : \neg Q(z) \Rightarrow R(z)]$

{ $\Rightarrow$-intro on (2) and (10): }

(11) $\forall_y[y \in \mathbb{N} : \neg P(y)] \Rightarrow \exists_z[z \in \mathbb{N} : \neg Q(z) \Rightarrow R(z)]$

{ $\Rightarrow$-intro on (1) and (11): }

(12) $\exists_x[x \in \mathbb{N} : P(x) \vee Q(x)] \Rightarrow (\forall_y[y \in \mathbb{N} : \neg P(y)] \Rightarrow \exists_z[z \in \mathbb{N} : \neg Q(z) \Rightarrow R(z)])$

Correction suggestions: A correct answer (3 points) gives a derivation showing that the formula is a tautology. One obvious variation on the above derivation is with an application of $\vee$-elim on (3) and (4) resulting in $P(x)$ followed by an application of $\neg$-elim on (5) and (6). Also, some rules may be applied in a slightly different order.

You may use the following scheme for awarding points to incomplete or partly correct solutions:

- for two correct applications of $\Rightarrow$-intro: 0.5 point;
- for a correct application of $\exists^*$-elim: 0.5 point;
- for a correct application of $\exists^*$-intro: 0.5 point;
- for a correct application of $\forall$-elim: 0.5 point;
- for a correct application of $\vee$-elim: 0.5 point;
- for a correct application of $\neg$-elim and `False`-elim: 0.5 point.

Let's not award more than 2 points for an incomplete or incorrect derivation.

Logically valid proofs that mix derivation-style reasoning with applications of calculation steps should also get at most 2 points.

If the application of the $\exists^*$ rule comes after the first use of the declared witness, but the derivation is otherwise correct, then you may also award 2 points.

Let's not be too strict about omission of hints: if some or all hints are missing (or misspelled), subtract (at most) 0.5 points in total.

We should be stricter about wrong applications of rules; it is worse to apply a rule incorrectly, than to not apply it at all. Subtract 0.5 point from the total number of points obtained according to the scheme above for every incorrect application of a rule. For instance, if the $\forall$-elim on the universal quantification in line (3) concludes $\neg P(y)$ for some undeclared variable y, then this incorrect application of $\forall$-elim does not yield 0.5 according to the scheme above, and, moreover, you also subtract 0.5 point from the total number of points awarded according to the scheme.

(2) 4. Prove that $|x-2| \geq -x+1$ for all $x \in \mathbb{R}$.

Solution: Let $x \in \mathbb{R}$. Then clearly $x \leq 2$ or $x \geq 2$. To prove that $|x-2| \geq -x+1$, we therefore distinguish these two cases.

If $x \leq 2$, then $x-2 \leq 0$, so $|x-2| = -x+2 > -x+1$. Hence, $|x-2| \geq -x+1$.

If $x \geq 2$, then $-x \leq -2$, and hence, since $x-2 \geq 0$, we have $|x-2| = x-2 \geq 0 > -1 \geq -2+1 \geq -x+1$. Hence, $|x-2| \geq -x+1$.

Correction suggestions: A correct answer (2 points) presents a complete and convincing proof of the statement. The proof may rely on standard properties of operations and predicates on real numbers.[1] If students use an alternative, but correct, definition of absolute value (e.g., $|x-2| = \max(x-2, -x+2)$) and thus, by using some obvious properties of max, manage to avoid case distinction, then this is fine.

Award 0.5 point for correctly dealing with the universal quantification (i.e., declaring $x \in \mathbb{R}$).

Award 1 point for correctly dealing with the definition of absolute value and the ensued logical treatment (probably most naturally by case distinction). If a good definition of $|x-2|$ is given, but then the argument stops, you may award 0.5 point instead of 1 for this aspect.

Award 0.5 point for mathematical core of the argument.

Some candidates will correctly distinguish cases, but then present the mathematical argument in the wrong direction (you have to read the mathematical core of the argument from bottom to top). Award 1 point in such cases.

[1] This is, of course, vague. It has been explained to students that the purpose of a proof is *not* to convince somebody else that the prover understands why the property holds, but really to give a good explanation to the other person why the statement holds. We judge proofs on their explanatory value and not just try to figure out whether the student himself or herself understands why the property is true.

(3) 5. Prove that every integer postage greater than 11 can be formed using only 3-cent and 7-cent stamps.

<u>*Solution:*</u> We prove with strong induction on integer postage $p > 11$ that p can be formed using only 3-cent and 7-cent stamps.

Let $p > 11$, and suppose that for all $11 < p' < p$, p' can be formed using only 3-cent and 7-cent stamps (the induction hypothesis). Clearly, if $p > 11$, then $p = 12$, $p = 13$, $p = 14$ or $p \geq 15$; we now distinguish these four cases below:

- If $p = 12$, then p can be formed using four 3-cent stamps.
- If $p = 13$, then p can be formed using two 3-cent stamps and one 7-cent stamp.
- If $p = 14$, then p can be formed using two 7-cent stamps.
- If $p \geq 15$, then $p - 3 > 11$ and $p - 3 < p$, so, by the induction hypothesis, $p - 3$ can be formed using only 3-cent and 7-cent stamps. We can use this formation of $p - 3$ to form p by simply adding an extra 3-cent stamp.

<u>*Correction suggestions:*</u> A correct answer, to be awarded with 3 points, consists of a complete and convincing proof of the statement. Attempts at proofs that do not use induction are likely to be incomplete (at least those that students may come up with), but one never knows ...

There are several ways to explain how an arbitrary postage $p > 11$ can be formed using only 3-cent and 7-cent stamps. (For instance, one could also have 7 base cases and then give a general argument for postages $p \geq 19$ that involves adding a 7 cent stamp to the postage $p - 7$.) It is important, here, that just explaining the core idea is not enough; induction should be used to 'formalise' the process.

The above proof uses strong induction. A proof with normal (mathematical) induction is also possible. Such a proof would then consider the basis case $p = 12$, and establish, in the step case, for arbitrary $p > 11$ that if p can be formed using only 3-cent and 7-cent stamps, then also $p + 1$ can be formed using only 3-cent and 7-cent stamps. Note that proving the property for $p + 1$ requires another case distinction: The induction hypothesis yields that there is an appropriate formation of p. If this formation has at least two 3-cent stamps, then two 3-cent stamps can be replaced by one 7-cent stamp, and otherwise there will be at least two 7-cent stamps that can be replaced by five 3-cent stamps.

The proof may also be presented in the form of a derivation (e.g., establishing the formula $\forall_p[p \in \mathbb{N} \wedge p > 11 : \exists_{k,\ell}[k, \ell \in \mathbb{N} : p = k \cdot 3 + \ell \cdot 7]]$). A mix of derivation-style and natural-language proof is also fine, provided that the logical reasoning is clear.

A proof by strong induction should include:

- The declaration of an arbitrary integer p with $p > 11$ (or $p \geq 12$) and a clear statement of the assumption that the property holds for all $11 < p' < p$ (the induction hypothesis). (In a derivation-style proof this will probably be kind of automatic: there should be two flags; one for the declaration of some arbitrary natural number, and the other stating the assumption that the property holds for all natural numbers smaller than the one declared.)

  Award 0.5 for an explicit declaration of integer p such that $p > 11$ and a correct statement of the induction hypothesis (i.e., the assumption that all integer postages p' with $11 < p' < p$ can be formed with 3-cent and 7-cent stamps only. (It need not be explicitly stated that the latter is the 'induction hypothesis'.)

  If the induction hypothesis is clearly stated, but it is about all p' less than some undeclared p, then you may still consider awarding 0.5 point.

- Either three or seven base cases (the number of base cases to be distinguished here depends on the method chosen in the induction step case).

  Award 1 point if all relevant base cases are mentioned and there is decent argument in each of the cases, explaining how the considered postage is to be formed.

  If the appropriate number of base cases is considered, but the arguments for postage formation are missing or unclear in those cases, then award 0.5 point.

  If the number of base cases considered does not correspond appropriately to the method chosen in the step case (e.g., only one base case whereas more is needed in view of the induction step), but there are reasonable arguments in these cases, then you may still award 0.5 point (depending on the quality of the arguments).

- Award 1.5 points in total for a separately treated step case, establishing the property for $p \geq 12$, as follows:

  It should be clear in the treatment of this case how the induction hypothesis is used. In particular, the proof should mention that $p-3$ (or $p-7$) is greater than 11 and less than p. And it should then be observed that according to the induction hypothesis the postage $p-5$ (or $p-9$) can be formed. If it is all nice and explicit, then award 1 point for this part. If you can recognise an application of the induction hypothesis, but it is implicit in the argument and it is also not explicitly said that $p-3$ (or $p-7$) indeed is greater than 11, less than p, then award 0.5 point for this part.

  Finally, it should be said that the postage p can be formed by adding an appropriate stamp to the postage $p-5$ (or $p-9$). This part is worth 0.5 point.

If the candidate gives a proof by normal induction, then award points as follows:

- There should be a separate basis case, arguing that the postage 35 can be formed. Award 0.5 point if the case $p = 12$ is distinguished and there is a decent argument that the postage can then be formed.

- There should be a clearly distinguishable step case, in which it is proved for all $p > 11$ that if p can be formed, then also $p+1$ can be formed. Award 1 point if the candidate deals with this goal in a logically valid way, i.e., there should be a declaration of an arbitrary integer p with $p > 11$ (or $p \geq 12$), a clear statement of the assumption that the property holds for this particular p (the induction hypothesis), and it should be clear that the candidate understands that then the goal is to establish that the property holds for $p + 1$. If these aspects are only partly recognisable, then award points as follows:

  Award 0.5 for an explicit declaration of integer p such that $p > 11$, and correctly stating the induction hypothesis (i.e., the assumption p can be formed with 3-cent and 7-cent stamps only). (It need not be explicitly said that the latter is the 'induction hypothesis'.) Note that the induction hypothesis should *not* state that the property holds *for all* p; it should really *declare* p and assume that the property holds for this particular p. So, you should not award points for the assumption that the property holds *for all* $p > 11$. If only the explicit declaration of p is forgotten, but there is no universal quantification, then you may still decide to award 0.5 point. If, on the other hand, there is only the declaration of $p > 11$, but you can not find the induction hypothesis stated anywhere explicitly, then do not award points for this part.

- To prove the property for $p+1$, there has to be a case distinction within the step case. It should be clear that all cases are covered, and in each of the cases there should be a decent argument why the property holds. Award 1.5 points for a decent case distinction and decent arguments in each of the cases.

Award 1 point if the candidate seems to realise that a case distinction is needed, there is a decent argument for at least one of the relevant cases, but there are missing cases, or in some of the cases the argument is invalid.

Award 0.5 point if the candidate does not seem to realise at all that a case distinction is needed and the provided argument for $p + 1$ only works in a subset of the relevant cases.

Try to evaluate any alternative presentation of a proof by induction that a candidate may come up with against the above principles: is there a clear statement of the induction hypothesis, is there a clear treatment of the base cases, and is there a decent reasoning for the step case. Try also to evaluate the logical correctness of the reasoning as a whole. If you feel that arguments are in the wrong order (e.g., the induction hypothesis is applied before it is stated), then you may subtract points, depending on the amount of unclarity.

(1) 6. (a) Show with a counterexample that the formula

$$\forall_Y [Y \in \mathcal{P}(\mathbb{N}) : \exists_X [X \in \mathcal{P}(\mathbb{Z}) : Y \subseteq F(X)]]$$

does not hold for all mappings $F : \mathbb{Z} \to \mathbb{N}$.

(1) (b) For a mapping $F : \mathbb{Z} \to \mathbb{N}$ and a set $Y \subseteq \mathbb{N}$, give the definition of the set $F^{\leftarrow}(Y)$, and prove that $F^{\leftarrow}(Y) \in \mathcal{P}(\mathbb{Z})$.

(2) (c) Prove that if $F : \mathbb{Z} \to \mathbb{N}$ is a surjection, then

$$\forall_Y [Y \in \mathcal{P}(\mathbb{N}) : \exists_X [X \in \mathcal{P}(\mathbb{Z}) : Y \subseteq F(X)]] \ .$$

Solution:

(a) Let, e.g., $F : \mathbb{Z} \to \mathbb{N}$ be the mapping defined, for all $x \in \mathbb{Z}$, by $F(x) = x^2$, and consider the set $Y = \{2\}$. To see that there does not exist $X \in \mathcal{P}(\mathbb{Z})$ such that $Y \subseteq F(X)$, let $X \in \mathcal{P}(\mathbb{Z})$ and suppose that $Y \subseteq F(X)$; we derive a contradiction.

From $X \in \mathcal{P}(\mathbb{Z})$ it follows, by the property of $\mathcal{P}$, that $X \subseteq \mathbb{Z}$. Since $2 \in Y$ it follows from $Y \subseteq F(X)$, by the property of $\subseteq$, that $2 \in F(X)$, and hence that there exists $x \in X$ such that $F(x) = x^2 = 2$. Moreover, from $X \in \mathcal{P}(\mathbb{Z})$ it follows that $X \subseteq \mathbb{Z}$, so $x \in \mathbb{Z}$. Now we have a contradiction, for there does not exist $x \in \mathbb{Z}$ such that $x^2 = 2$.

(b) For a mapping $F : \mathbb{Z} \to \mathbb{N}$, the set $F^{\leftarrow}(Y)$ is defined by

$$F^{\leftarrow}(Y) = \{x \in \mathbb{Z} \mid F(x) \in Y\} \ .$$

To prove that $F^{\leftarrow}(Y) \in \mathcal{P}(\mathbb{Z})$, note that from the definition of $F^{\leftarrow}(Y)$ it follows by the property of $\in$ that for all x whenever $x \in F^{\leftarrow}(Y)$ then also $x \in \mathbb{Z}$, so, by the property of $\subseteq$, $F^{\leftarrow}(Y) \subseteq \mathbb{Z}$ and therefore, by the property of $\mathcal{P}$, $F^{\leftarrow}(Y) \in \mathcal{P}(\mathbb{Z})$.

(c) Let $F : \mathbb{Z} \to \mathbb{N}$ be a surjection, and let $Y \in \mathcal{P}(\mathbb{N})$; to prove that there exists $X \in \mathcal{P}(\mathbb{Z})$ such that $Y \subseteq F(X)$, we need to establish that $F^{\leftarrow}(Y) \in \mathcal{P}(\mathbb{Z})$ and $Y \subseteq F(F^{\leftarrow}(Y))$.

That $F^{\leftarrow}(Y) \in \mathcal{P}(\mathbb{Z})$ has already been established in part (b).

To establish that $Y \subseteq F(F^{\leftarrow}(Y))$, let $y \in Y$; it remains to prove that then also $y \in F(F({}^{\leftarrow}(Y)))$. From $y \in Y$ it follows, since F is a surjection, that there exists $x \in \mathbb{Z}$ such that $F(x) = y$. Then, from $F(x) \in Y$, it follows, by the property of source, that $x \in F^{\leftarrow}(Y)$, and hence, by the property of image, that $F(x) = y \in F(F^{\leftarrow}(Y))$. We conclude, using $F(x) = y$ again, that $y \in F(F^{\leftarrow}(Y))$.

Correction suggestions:

(a) A correct answer (1 point) defines an appropriate mapping $F : \mathbb{Z} \to \mathbb{N}$ and a set $Y \subseteq \mathbb{N}$ and argues that there does not exist $X \in \mathcal{P}(\mathbb{Z})$ such that $Y \subseteq F(X)$.

For a correct concrete definition of a mapping $F : \mathbb{Z} \to \mathbb{N}$ and a set Y for which there does not exist $X \in \mathcal{P}(\mathbb{Z})$ such that $Y \subseteq F(X)$, award 0.5 point.

An argument that the given F and Y indeed refute the statement gives the other 0.5 point. The argument need not be as elaborate as above; just some informal explanation that from some natural number $y \in Y$ there does not exist an integer $x \in X$ suffices.

(b) A correct answer (1 point) provides the definition of $F^{\leftarrow}(Y)$ and proves that $F^{\leftarrow}(Y) \in \mathcal{P}(\mathbb{Z})$.

Award 0.5 point for the definition of $F^{\leftarrow}(Y)$.

Award 0.5 point if the candidate also manages to argue in a convincing way that $F^{\leftarrow}(Y) \in \mathcal{P}(\mathbb{Z})$. (I propose to be convinced when the candidate simply observes that the elements of $F^{\leftarrow}(Y)$ are integers according to the definition and hence $F^{\leftarrow}(Y)$ is a subset $\mathbb{Z}$, without properly referring explicitly to the properties and definitions of $\in$, $\subseteq$, and $\mathcal{P}$.)

(c) A correct answer (2 points) proves that there exists $X \in \mathcal{P}(\mathbb{Z})$ such that $Y \subseteq F(X)$ if F is a surjection. The proof should be based on the properties and definitions of sets and mappings as given in the book.

Award points different aspects of the answer as follows:

- Award 0.5 point if the candidate shows that he or she understands the definition of surjection (either because the formula is mentioned, or because the property of surjection is used correctly in the proof).
- Award 0.5 point if the candidate correctly applies the definition of $\subseteq$ (in order to prove that $Y \subseteq F(X)$, it suffices to prove that every element $y \in Y$ is an element of $F(X)$).
- Award 0.5 point for understanding and correctly using the properties of image and source of mappings.
- Award 0.5 point for correct formulation of the logical reasoning involved.

7. We define a binary relation R on $\mathbb{N}^+$, the set of all positive natural numbers, by

$$k \; R \; \ell \text{ if, and only if, there exists } c \in \mathbb{N}^+ \text{ such that } c \text{ is a multiple of 3 and } \ell = c \cdot k.$$

(2) (a) Prove that R is transitive.

Let $V = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

(1) (b) Make a Hasse diagram of $\langle V, R \rangle$.

(1) (c) Define an *infinite* set $W \subseteq \mathbb{N}^+$ such that $\langle W, R \rangle$ is linear.

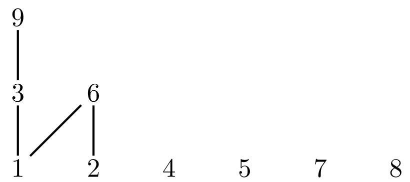
<u>*Solution:*</u>

(a) To prove that R is transitive, let $k, \ell, m \in \mathbb{N}^+$, and suppose that $k \; R \; \ell$ and $\ell \; R \; m$; we need to establish that $k \; R \; m$.

From $k \; R \; \ell$ and $\ell \; R \; m$, it follows, according to the definition of R, there exist $c, d \in \mathbb{N}^+$ such that c and d both multiples of 3, and $m = c \cdot \ell$ and $\ell = d \cdot k$. So $m = c \cdot \ell = c \cdot (d \cdot k) = (c \cdot d) \cdot k$. Moreover, since $c, d \in \mathbb{N}^+$ and c and d are both multiples of 3, we also have that $c \cdot d \in \mathbb{N}^+$ and $c \cdot d$ is a multiple of 3.

Hence, we may conclude, according to the definition of R, that $k \; R \; m$.

(b)



(c) If $W = \{3^n \mid n \in \mathbb{N}^+\}$, then W is infinite and $\langle W, R \rangle$ is linear.

<u>*Correction suggestions:*</u>

(a) A correct answer (2 points) proves that R is transitive.

Award 1 point if the candidate shows that he or she knows when a relation is transitive and correctly sets up the proof, with the declaration of three arbitrary positive natural numbers, assuming that the first is related to the second, and the second is related to the third, and stating that the goal is to establish that the first is related to the third. If the candidate only reproduces the formula for 'R is transitive' (and nothing else), then award 0.5 point.

Award 0.5 point for correctly dealing with the definition of R (in particular, declaring two natural numbers greater or equal 2).

Award 0.5 point for the mathematical part of the argument.

(b) A correct answer (1 point) gives the Hasse diagram. The Hasse diagram should, of course, not have fewer or more connections between the elements. The relative positioning of connected elements (as above) should correctly indicate the 'direction' of the relation. Alternatively, the direction may be indicated by an arrow instead of a connection without arrowheads, and then the relative positioning is not important.

You may still award 0.5 point if the Hasse diagram is not correct, but reasonably close to the correct one.

(c) A correct answer (1 point) gives an infinite set of positive natural numbers on which R is linear.

Award 0.5 point if a reasonably sized finite subset of positive natural numbers of, say, at least 3 elements is given on which R is linear.

Also award 0.5 if the candidate shows that he or she understands the definition of linearity.