Mappings

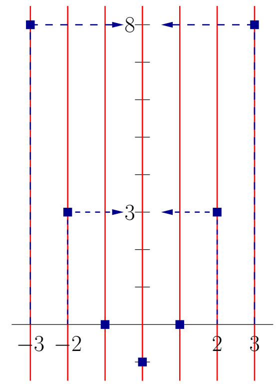
Lectures 9 and 10 (Chapter 18)

Example

Let R be a binary relation on $\mathbb{Z}$ defined by

$$x \; R \; y \text{ if } y = x^2 - 1 \; .$$

On every vertical line exactly 1 point.



Mapping (1)

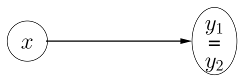
The relation R between A and B is a mapping (function) if

$$\forall_x [x \in A : \underbrace{\exists^1_y [y \in B : x \; R \; y]}]$$

at least one: $\exists_y [y \in B : x \; R \; y]$

$\wedge$

at most one: $\forall_{y_1, y_2} [y_1, y_2 \in B : (x \; R \; y_1 \wedge x \; R \; y_2) \Rightarrow y_1 = y_2]$



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Exercise

Determine if the following relations are mappings:

1. $R_1 \subseteq \mathbb{R} \times \mathbb{R}$ defined by $x \; R_1 \; y$ iff $y = x^2$ — Yes
2. $R_2 \subseteq \mathbb{R} \times \mathbb{R}$ defined by $x \; R_2 \; y$ iff $y^2 = x$ — No
3. $R_3 \subseteq \mathbb{R}^+ \times \mathbb{R}^+$ defined by $x \; R_3 \; y$ iff $y^2 = x$ — Yes
4. $R_4 \subseteq \mathbb{R} \times \mathbb{R}$ defined by $x \; R_4 \; y$ iff $y = 2x \vee y = 3x$ — No

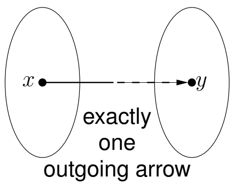
Mapping (2)

$F : A \to B$ means "$F \subseteq A \times B$ and F is a mapping (function)"

A: domain of F
B: range of F

We usually write $y = F(x)$ instead of $x \; F \; y$

> Property of mapping $F : A \to B$:
> $\forall_x[x \in A : \exists^1_y[y \in B : F(x) = y]]$



so

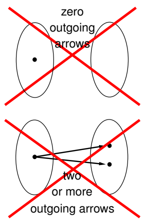
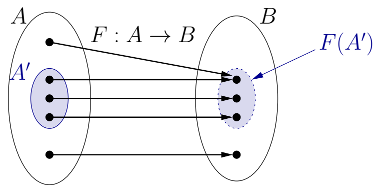


Image (1)



The image $F(A')$ of A' under F is the set of all end-points in B of arrows starting in A'.

$$F(A') \stackrel{\text{def}}{=} \{b \in B \mid \exists_x[x \in A' : F(x) = b]\}$$

Example:
$F : \mathbb{Z} \to \mathbb{Z}$ with $F(x) = x^2 - 1$

Then: $F(\{0, 2\}) = \{-1, 3\}$
$F(\emptyset) = \emptyset$
$F(\mathbb{Z}) = \{-1, 0, 3, 8, 15, 24, \dots\}$

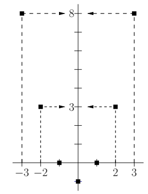
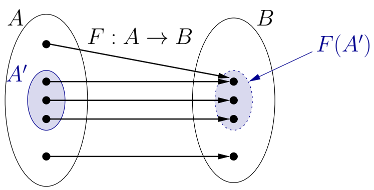


Image (2)



> Properties of 'image':
> $x \in A' \stackrel{val}{\models} F(x) \in F(A')$
> $y \in F(A') \stackrel{val}{=\!=} \exists_x[x \in A' : F(x) = y]$

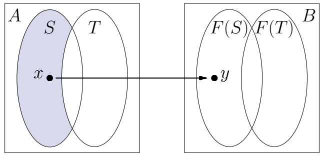
NB: $x \in A' \stackrel{val}{\not=} F(x) \in F(A')$; it may happen that $F(x) \in F(A')$ and at the same time $x \notin A'$. (*Counterexample* on p. 264 of the book.)

Example

Let $F : A \to B$ be a mapping, and let $S, T \subseteq A$.
Prove that $F(S) \backslash F(T) \subseteq F(S \backslash T)$.

[Proof on blackboard. (Also available as detailed example of the construction of the proof of a property involving sets and mappings from Course Material section of the website)]

Example: $F(S) \backslash F(T) \subseteq F(S \backslash T)$



(1) **var** y; $y \in F(S) \setminus F(T)$

(2) $y \in F(S) \wedge \neg(y \in F(T))$

(3) $\exists_x[x \in S : F(x) = y]$ (Prop. image)

(4) Pick an x with $x \in S$ and $F(x) = y$

(5) $x \in T$

(6) $F(x) \in F(T)$ (Prop. image)

(7) $y \in F(T)$

(8) `False`

(9) $\neg(x \in T)$

(10) $x \in S \wedge \neg(x \in T)$

(11) $x \in S \setminus T$ (Prop. \)

(12) $\exists_x[x \in S \setminus T : F(x) = y]$ (Prop. image)

(13) $y \in F(S \setminus T)$

(14) $F(S) \backslash F(T) \subseteq F(S \backslash T)$

Properties of 'image':
$x \in A' \stackrel{val}{\Longrightarrow} F(x) \in F(A')$
$y \in F(A') \stackrel{val}{=\!=} \exists_x[x \in A' : F(x) = y]$

NB: the other direction does not hold.
Exercise: give a counterexample.

Example: $F(S) \backslash F(T) \subseteq F(S \backslash T)$

Proof:

According to the property of $\subseteq$, we need to prove that all elements of $F(S) \backslash F(T)$ are also elements of $F(S \backslash T)$.

Let $y \in F(S) \backslash F(T)$; we need to establish that $y \in F(S \backslash T)$. To this end, it suffices, by the property of image, to prove the existence of $x \in S \backslash T$ such that $F(x) = y$.

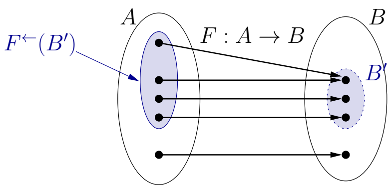
Note that, from $y \in F(S) \backslash F(T)$ it follows, by the property of $\backslash$, that $y \in F(S)$ and $y \notin F(T)$. So, by the property of image, there exists $x \in S$ such that $F(x) = y$.

It therefore remains to prove that $x \in S \backslash T$, which, according to the property of $\backslash$ and since $x \in S$, amounts to proving that $x \notin T$.

To prove $x \notin T$, we suppose that $x \in T$, and derive a contradiction. From $x \in T$ it follows, by the property of image, that $F(x) \in F(T)$. Hence, since $F(x) = y$, we have that $y \in F(T)$.
Since also $y \notin F(T)$ (see above), we have thus arrived at a contradiction. □

Source (1)

The source $F^{\leftarrow}(B')$ of B' is the set of all starting points in A of arrows with their end-point in B'.

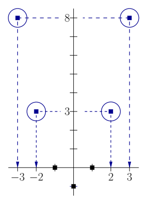


$$F^{\leftarrow}(B') \stackrel{\text{def}}{=} \{a \in A \mid F(a) \in B'\}$$

Example:

$F : \mathbb{Z} \to \mathbb{Z}$ with $F(x) = x^2 - 1$

Then: $F^{\leftarrow}(\{1, \ldots, 10\}) = \{-3, -2, 2, 3\}$



Source (2)

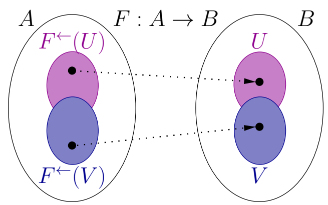
Property of 'source':
$x \in F^{\leftarrow}(B') \stackrel{val}{=\!=} F(x) \in B'$

Example:

Let $F : A \to B$ and $U, V \subseteq B$.
Prove that $F^{\leftarrow}(U) \cup F^{\leftarrow}(V) = F^{\leftarrow}(U \cup V)$.

For all $x \in \mathcal{U}$:

$x \in F^{\leftarrow}(U) \cup F^{\leftarrow}(V)$
$\stackrel{val}{=\!=}$ { Property of $\cup$ }
$x \in F^{\leftarrow}(U) \vee x \in F^{\leftarrow}(V)$
$\stackrel{val}{=\!=}$ { Property of source }
$F(x) \in U \vee F(x) \in V$
$\stackrel{val}{=\!=}$ { Property of $\cup$ }
$F(x) \in U \cup V$
$\stackrel{val}{=\!=}$ { Property of source }
$x \in F^{\leftarrow}(U \cup V)$



Source (3)

Lemma

If $F : A \to B$ en $A' \subseteq A$, then $F^{\leftarrow}(F(A')) \supseteq A'$.

Proof: see book pp. 264–265.

NB: A' can be really smaller than $F^{\leftarrow}(F(A'))$:

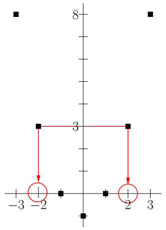
$F : \mathbb{Z} \to \mathbb{Z}$ with $F(x) = x^2 - 1$, $A' = \{2\}$.

Then

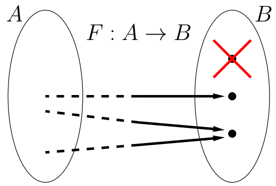
$$F(\{2\}) = \{3\}$$

and

$$F^{\leftarrow}(F(\{2\})) = \{-2, 2\} \neq \{2\}.$$



Surjection



F is a **surjection** if every $b \in B$ has **at least one** incoming arrow.

Property of surjection:
$\forall_y[y \in B : \exists_x[x \in A : F(x) = y]]$

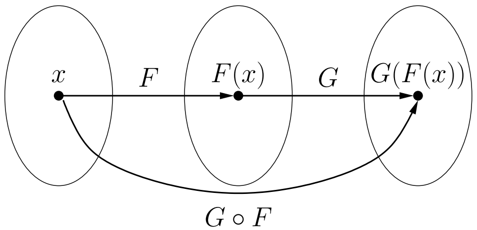
Examples:

- $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x^2$ **is not** a surjection.
- $f : \mathbb{R} \to \mathbb{R}^+ \cup \{0\}$ with $f(x) = x^2$ **is** a surjection.

Composed mapping

Let A, B and C be sets.
Let $F : A \to B$ and $G : B \to C$ be mappings.
The **composed mapping** $G \circ F$ (pronounce: 'G after F') is defined by

$$(G \circ F)(x) = G(F(x)) \qquad \text{for all } x \in A \ .$$



Example

Lemma

If $F : A \to B$ and $G : B \to C$ are both surjections, then $G \circ F$ is a surjection too.

Proof:

var z; $z \in C$

$\exists_y[y \in B : G(y) = z]$ (∀-elim with $z \in C$ on 'G is a surjection')

Pick a $y \in B$ with $G(y) = z$

$\exists_x[x \in A : F(x) = y]$ (∀-elim with $y \in B$ op 'F is a surjection')

Pick an $x \in A$ with $F(x) = y$

$(G \circ F)(x) = G(F(x)) = G(y) = z$

$\exists_x[x \in A : (G \circ F)(x) = z]$

$\forall_z[z \in C : \exists_x[x \in A : (G \circ F)(x) = z]]$

Example

Lemma

If $F : A \to B$ and $G : B \to C$ are both surjections, then $G \circ F$ is a surjection too.

Proof:

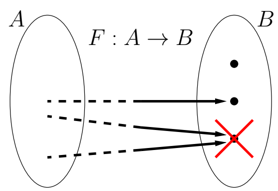
To prove that $G \circ F$ is a surjection, according to the definition of surjection we need to establish that for all $z \in C$ there exists $x \in A$ such that $(G \circ F)(x) = z$.

To this end, let $z \in C$. Since G is a surjection, there exists $y \in B$ such that $G(y) = z$, and hence, since F is a surjection, there exists $x \in A$ such that $F(x) = y$.

It follows, using the definition of $\circ$, $F(x) = y$ and $G(y) = z$, that

$$(G \circ F)(x) = G(F(x)) = G(y) = z \ . \qquad \square$$

Injection



F is an **injection** if every $b \in B$ has **at most one** incoming arrow.

Property of injection:
$\forall_{x_1,x_2}[x_1, x_2 \in A : F(x_1) = F(x_2) \Rightarrow x_1 = x_2]$

Example:

- $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x^2$ **is not** an injection.
- $f : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$ with $f(x) = x^2$ **is** an injection.

Example

Lemma

If $F : A \to B$ and $G : B \to C$ are both injections, then so is $G \circ F$.

Proof:

var $x_1, x_2;\ x_1, x_2 \in A$

$(G \circ F)(x_1) = (G \circ F)(x_2)$

$F(x_1), F(x_2) \in B$	($\forall$-elim+$\exists^*$-elim on Prop. mapping+Leibniz)
$G(F(x_1)) = G(F(x_2))$	(Def. of composed mapping $G \circ F$)
$G(F(x_1)) = G(F(x_2)) \Rightarrow F(x_1) = F(x_2)$	($\forall$-elim on 'G is an injection')
$F(x_1) = F(x_2)$	
$F(x_1) = F(x_2) \Rightarrow x_1 = x_2$	($\forall$-elim on 'F is an injection')
$x_1 = x_2$	

$\forall_{x_1,x_2}[x_1, x_2 \in A : (G \circ F)(x_1) = (G \circ F)(x_2) \Rightarrow x_1 = x_2]$

Example

Lemma

If $F : A \to B$ and $G : B \to C$ are both injections, then so is $G \circ F$.

Proof:

To prove that $(G \circ F)$ is an injection, according to the definition of injection we need to establish for all $x_1, x_2 \in A$ that $(G \circ F)(x_1) = (G \circ F)(x_2)$ implies $x_1 = x_2$.

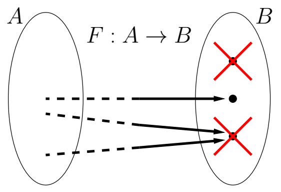
So, let $x_1, x_2 \in A$ and suppose that $(G \circ F)(x_1) = (G \circ F)(x_2)$; we show that $x_1 = x_2$.

From $(G \circ F)(x_1) = (G \circ F)(x_2)$ it follows, by the definition of $\circ$, that $G(F(x_1)) = G(F(x_2))$.

Hence, since G is an injection, we have that $F(x_1) = F(x_2)$.

Therefore, since F is an injection, we have that $x_1 = x_2$. $\square$

Bijection



F is a **bijection** if every $b \in B$ has **exactly one** incoming arrow.

Property of bijection:
$\forall_y [y \in B : \exists^1_x [x \in A : F(x) = y]]$

usually: $\forall_y [y \in B : \exists_x [x \in A : F(x) = y]]$ (surjection)

$\wedge$

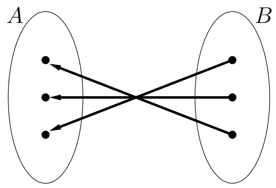
$\forall_{x_1, x_2} [x_1, x_2 \in A : F(x_1) = F(x_2) \Rightarrow x_1 = x_2]$ (injection)

Bijection (examples)

1. $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x^2$ **is not a bijection**.
   $f : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$ with $f(x) = x^2$ **is a bijection**.
2. $F : \mathbb{N} \to \{n \in \mathbb{N} \mid n \text{ is even}\}$ with $F(n) = 2n$ is a bijection.

Inverse

Suppose $F : A \to B$ is a bijection.
Then there exists a one-to-one correspondence between elements of A and B.
We can now invert all arrows.



Thus, we get the **inverse mapping** $F^{-1} : B \to A$ of F.

Property of inverse $F^{-1} : B \to A$ of bijection $F : A \to B$:
$F(x) = y \overset{val}{\equiv} F^{-1}(y) = x$

NB: F^{-1} is again a 'true' mapping.

Inverse (examples)

1. Let $f : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$ be the bijection defined by
   $$f(x) = x^2 \ .$$
   Then $f^{-1} : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$ is the bijection defined by
   $$f^{-1}(x) = \sqrt{x} \ .$$
2. If $F : A \to B$ is a bijection, then
   $F^{-1} \circ F : A \to A$ is the **indentity on** A
   (i.e., $(F^{-1} \circ F)(x) = x$ for all $x \in A$);
   $F \circ F^{-1} : B \to B$ the **identity on** B