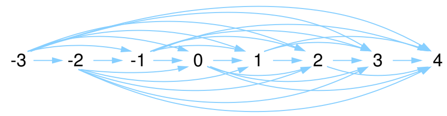
Ordered sets

Lecture 13 (Chapter 20)

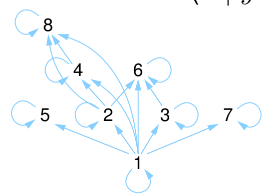
October 19, 2013

Ordering (examples)

1. The binary relation $<$ *orders* the elements of $\{-3, \ldots, 4\} \subseteq \mathbb{Z}$:



2. The binary relation $\mid$ *orders* the elements of $\{1, \ldots, 8\} \subseteq \mathbb{N}^+$ ($x \mid y$ iff $\exists_k[k \in \mathbb{N}^+ : y = k \cdot x]$):

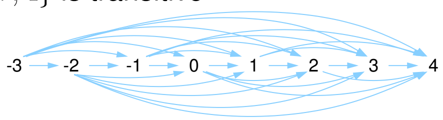


Ordering: transitivity

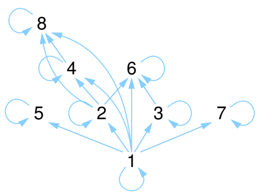
Both relations are transitive: $\forall_{x,y,z}[x, y, z \in A : x\ R\ y \wedge y\ R\ z \Rightarrow x\ R\ z]$ ($A = \{-3, \ldots, 4\}$ or $A = \{1, \ldots, 8\}$, $R = <$ or $R = \mid$).

Examples

1. $<$ on $\{-3, \ldots, 4\}$ is transitive



2. $\mid$ on $\{1, \ldots, 8\}$ is transitive

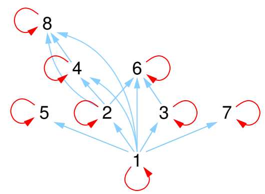


Ordering: reflexive or irreflexive?

Examples

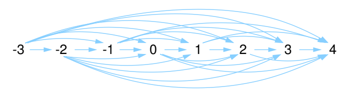
1. The binary relation $\mid$ on $\{1, \ldots, 8\}$ is reflexive:

$$\forall_x[x \in \{1, \ldots, 8\} : x \mid x]$$



2. The binary relation $<$ on $\{-3, \ldots, 4\}$ is irreflexive:

$$\forall_x[x \in \{-3, \ldots, 4\} : \neg(x < x)]$$



We consider irreflexive and reflexive orderings.

Irreflexive is stronger than *Not Reflexive*

irreflexive: $\forall_x[x \in A : \neg(x\ R\ x)]$

not reflexive: $\neg\forall_x[x \in A : x\ R\ x]$

R *not reflexive* does not imply R *irreflexive*:

Define R on $\mathbb{Z}$ for all $x \in \mathbb{Z}$ by:

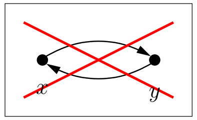
$$x\ R\ y \text{ if, and only if, } y = 3x\ .$$

Then:

- *R is not reflexive, for we have $\neg(1\ R\ 1)$, but*
- *R is not irreflexive either, for we have $0\ R\ 0$.*

R *irreflexive* does imply R *not reflexive*.

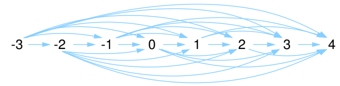
Ordering: (strict) antisymmetry



$$\forall_{x,y}[x, y \in A : x\ R\ y \wedge y\ R\ x \Rightarrow \text{NO}]\ .$$

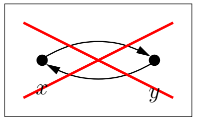
Example 1

The binary relation $<$ on $\{-3, \ldots, 4\}$



is strictly antisymmetric: $\forall_{x,y}[x, y \in A : \underbrace{x\ R\ y \wedge y\ R\ x \Rightarrow \texttt{False}}_{\neg(x\ R\ y\ \wedge\ y\ R\ x)}]$

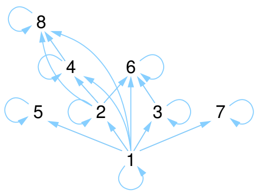
Ordering: (strict) antisymmetry



$$\forall_{x,y}[x, y \in A : x\ R\ y \wedge y\ R\ x \Rightarrow \text{NO}]\ .$$

Example 2

The binary relation | on $\{1, \ldots, 8\}$



is antisymmetric: $\forall_{x,y}[x, y \in A : x\ R\ y \wedge y\ R\ x \Rightarrow x = y]$

Reflexive orderings, Irreflexive orderings

I. Reflexive ordering $\langle A, R\rangle$:
- I.1 reflexive: $\forall_x[x \in A : x\ R\ x]$
- I.2 antisymmetric: $\forall_{x,y}[x, y \in A : x\ R\ y \wedge y\ R\ x \Rightarrow x = y]$
- I.3 transitive: $\forall_{x,y,z}[x, y, z \in A : x\ R\ y \wedge y\ R\ z \Rightarrow x\ R\ z]$

II. Irreflexive ordering $\langle A, R\rangle$:
- II.1 irreflexive: $\forall_x[x \in A : \neg(x\ R\ x)]$
- II.2 strictly antisymmetric: $\forall_{x,y}[x, y \in A : \neg(x\ R\ y \wedge y\ R\ x)]$
- II.3 transitive: $\forall_{x,y,z}[x, y, z \in A : x\ R\ y \wedge y\ R\ z \Rightarrow x\ R\ z]$

NB: II.2 follows from II.1+II.3 (see Ex. 20.2); we may therefore omit requirement II.2 from the definition.

Whenever you need to prove that a relation is an irreflexive ordering, you only need to prove that it is irreflexive and transitive; you may omit the proof of strict antisymmetry.

Examples

- $\langle \mathbb{N}, \leq \rangle$, $\langle \mathbb{Z}, \leq \rangle$ and $\langle \mathbb{R}, \leq \rangle$ are reflexive orderings
- $\langle \mathbb{N}, < \rangle$, $\langle \mathbb{Z}, < \rangle$ and $\langle \mathbb{R}, < \rangle$ are irreflexive orderings
- $\langle \mathbb{N}^+, | \rangle$ is a reflexive ordering
  [For proof of reflexivity: see the book; we prove antisymmetry; do transitivity yourself.]
- $\langle \mathcal{P}(A), \subseteq \rangle$ is a reflexive ordering
  [Proof on next slide.]

$\langle \mathcal{P}(A), \subseteq \rangle$ is a reflexive ordering

Proof:
We need to prove that $\subseteq$ on $\mathcal{P}(A)$ is (1) reflexive, (2) antisymmetric and (3) transitive:

1. For arbitrary $X \in \mathcal{P}(A)$, $X \subseteq X$ follows directly from the definition of $\subseteq$.
   Hence, $\subseteq$ on $\mathcal{P}(A)$ is reflexive.
2. Let $X, Y \in \mathcal{P}(A)$ and suppose that $X \subseteq Y$ and $Y \subseteq X$; then $X = Y$ follows directly from the definition of $=$ on sets.
   Hence, $\subseteq$ on $\mathcal{P}(A)$ is antisymmetric.
3. Let $X, Y, Z \in \mathcal{P}(A)$ and suppose that $X \subseteq Y$ and $Y \subseteq Z$.
   By the Property of $\subseteq$, it holds for every element $x \in X$ that $x \in Y$, so, again by the Property of $\subseteq$, we get $x \in Z$.
   It follows that $X \subseteq Z$.
   Hence, $\subseteq$ on $\mathcal{P}(A)$ is transitive.

Comparable

Let $\langle A, R \rangle$ be a reflexive or irreflexive ordering, and let $x, y \in A$.

Then there are three possibilities:

$x\ R\ y$?	yes/no
$y\ R\ x$?	yes/no
$x = y$?	yes/no

x and y are comparable if we get at least one time ‘yes’.

x and y are incomparable if we get three times ‘no’.

Comparable (example 1)

- $\langle \mathbb{N}^+, | \rangle$

x	y	comparable?
3	15	✓
15	3	✓
3	3	✓
3	7	×
7	3	×

} sometimes ‘yes’, sometimes ‘no’

Comparable (example 2)

- $\langle \mathbb{Z}, \leq \rangle$

x	y	comparable?
3	15	✓
15	3	✓
3	3	✓
3	7	✓
7	3	✓

always 'yes'!

Linear orderings (definition)

A (reflexive or irreflexive) ordering $\langle A, R \rangle$ is linear if *every two elements are comparable*, i.e.,

$$\forall_{x,y}[x, y \in A : x \; R \; y \vee y \; R \; x \underbrace{\vee \; x = y}]$$

may be omitted if R is reflexive

Examples

- $\langle \mathbb{Z}, \leq \rangle$ is a linear reflexive ordering and $\langle \mathbb{Z}, < \rangle$ is a linear irreflexive ordering
- $\langle \mathbb{N}^+, | \rangle$ is *not* linear

NB: If $\langle A, R \rangle$ is a *linear* ordering, then the elements of A can be arranged on a straight line such that $x \in A$ is left of $y \in A$ iff xRy.

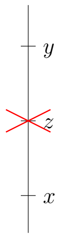
[Example: recall the arrangement of $\langle \mathbb{Z}, < \rangle$ on slide 2.]

Direct successors/predecessors

1. Let $\langle A, R \rangle$ be an irreflexive ordering

   $y \in A$ is a direct successor of $x \in A$ if

   $$x \; R \; y \wedge \neg \exists_z [z \in A : x \; R \; z \wedge z \; R \; y]$$



2. Let $\langle A, R \rangle$ be a reflexive ordering

   $y \in A$ is a direct successor of $x \in A$ if

   $$x \; R \; y \wedge x \neq y \wedge \neg \exists_z [z \in A \wedge z \neq x \wedge z \neq y : x \; R \; z \wedge z \; R \; y]$$

Direct successors

Examples

- In $\langle \mathbb{N}, < \rangle$:
  - every $n \in \mathbb{N}$ has a direct successor $n + 1$, and
  - every $n \in \mathbb{N} \backslash \{0\}$ has a direct predecessor $n - 1$.
- In $\langle \mathbb{R}, < \rangle$ there are no direct successors.
- In $\langle \mathbb{N}^+, | \rangle$, every $n \in \mathbb{N}$ has infinitely many direct successors.

  direct successors of 3: $6, 9, 15, \ldots$

  examples of numbers that are not direct successors of 3:
  - 3 (because direct successor should be distinct),
  - 12 (because $3 \mid 6 \mid 12$),
  - 18 (because $3 \mid 6 \mid 18$ and also $3 \mid 9 \mid 18$),
  - 4 (because $3 \nmid 4$).

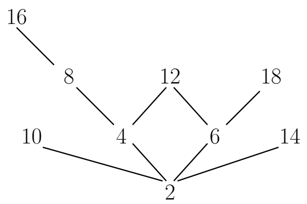
  NB: In $\langle \mathbb{N}^+, | \rangle$, the number n is a direct successor of m if $n = p \cdot m$ with p prime.

Hasse diagrams

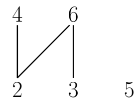
Orderings with direct predecessors/successors we can arrange as Hasse diagrams.

A Hasse diagram has a connection between x and a y above it if, and only if, y is a direct successor of x.

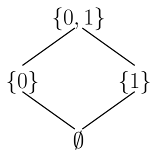
- $\langle\{2,4,6,8,10,12,14,16,18\},|\rangle$



- $\langle\{2,3,4,5,6\},|\rangle$



- $\langle\mathcal{P}(\{0,1\}),\subseteq\rangle$



Maximum/minimum

Let $\langle A,R\rangle$ be an ordering, and let $A' \subseteq A$.

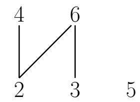
$m \in A'$ is the maximum of A' if

$$\begin{cases} \forall_x[x \in A' \wedge x \neq m : x\ R\ m] & \text{for an irreflexive ordering} \\ \forall_x[x \in A' : x\ R\ m] & \text{for a reflexive ordering} \end{cases}.$$

$m \in A'$ is the minimum of A': analogous

Examples:

- Consider $\langle\{2,3,4,5,6\},|\rangle$
  - maximum $\{2,3,6\}$: 6
  - minimum $\{2,3,6\}$: no minimum
  - maximum $\{2,3,5,6\}$: no maximum
  - minimum $\{2,3,5,6\}$: no minimum



Maximum/minimum

Let $\langle A,R\rangle$ be an ordering, and let $A' \subseteq A$.

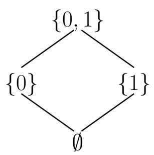
$m \in A'$ is the maximum of A' if

$$\begin{cases} \forall_x[x \in A' \wedge x \neq m : x\ R\ m] & \text{for an irreflexive ordering} \\ \forall_x[x \in A' : x\ R\ m] & \text{for a reflexive ordering} \end{cases}.$$

$m \in A'$ is the minimum of A': analogous

Examples:

- Consider $\langle\mathcal{P}(\{0,1\}),\subseteq\rangle$
  - maximum $\mathcal{P}(\{0,1\})$: $\{0,1\}$
  - minimum $\mathcal{P}(\{0,1\})$: $\emptyset$



Maximum/minimum

Let $\langle A,R\rangle$ be an ordering, and let $A' \subseteq A$.

$m \in A'$ is the maximum of A' if

$$\begin{cases} \forall_x[x \in A' \wedge x \neq m : x\ R\ m] & \text{for an irreflexive ordering} \\ \forall_x[x \in A' : x\ R\ m] & \text{for a reflexive ordering} \end{cases}.$$

$m \in A'$ is the minimum of A': analogous

Examples:

- Consider $\langle\mathbb{Z},\leq\rangle$
  - maximum $\mathbb{N}$: no maximum
  - minimum $\mathbb{N}$: 0
  - maximum $\mathbb{Z}$: no maximum
  - minimum $\mathbb{Z}$: no minimum

Exercise (maximum)

Determine whether the following sets have a maximum/minimum in the given ordering, and if so, then give the maximum/minimum.

- $\langle \mathbb{R}, \leq \rangle$
  $\{0, \ldots, 10\}$: maximum 10, minimum 0
  $\{x \in \mathbb{N} \mid x$ is even$\}$: no maximum, minimum 0
  $\langle 0, 1 \rangle$: no maximum, no minimum
- $\langle \mathcal{P}(\mathbb{N}), \subseteq \rangle$
  $\mathcal{P}(\{0, 1\})$: maximum $\{0, 1\}$, minimum $\emptyset$
  $\{\{x \in \mathbb{N} \mid x$ is even$\}\}$:
  $\{x \in \mathbb{N} \mid x$ is even$\}$ maximum and minimum!
- $\langle \mathbb{N}^{+}, | \rangle$
  $\{2, 3, 4, 5, 6\}$: no maximum, no minimum
  $\{2, 3, 4, 6, 8, 24\}$: maximum 24, no minimum

Maximal/minimal elements

Let $\langle A, R \rangle$ be an ordering, and let $A' \subseteq A$.

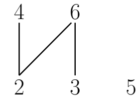
$m \in A'$ is a maximal element of A' if

$$\begin{cases} \forall_x [x \in A' : \neg(m\ R\ x)] & \text{for an irreflexive ordering} \\ \forall_x [x \in A' : m\ R\ x \Rightarrow m = x] & \text{for a reflexive ordering} \end{cases} .$$

$m \in A'$ is a minimal element of A': analogous

Examples:

- Consider $\langle \{2, 3, 4, 5, 6\}, | \rangle$
  max. elements $\{2, 3, 5, 6\}$: $5, 6$
  min. elements $\{2, 3, 5, 6\}$: $2, 3, 5$



Maximal/minimal elements

Let $\langle A, R \rangle$ be an ordering, and let $A' \subseteq A$.

$m \in A'$ is a maximal element of A' if

$$\begin{cases} \forall_x [x \in A' : \neg(m\ R\ x)] & \text{for an irreflexive ordering} \\ \forall_x [x \in A' : m\ R\ x \Rightarrow m = x] & \text{for a reflexive ordering} \end{cases} .$$

$m \in A'$ is a minimal element of A': analogous

Examples:

- Consider $\langle \mathbb{Z}, < \rangle$
  max. elements of $\mathbb{N}$: no maximal elements
  min. elements of $\mathbb{N}$: 0
  max. elements of $\mathbb{Z}$: no maximal elements
  min. elements of $\mathbb{Z}$: no minimal elements

Maximum and maximal element

Theorem:
The maximum, whenever it exists, is a maximal element.

Proof:
We prove the theorem for a reflexive ordering $\langle A, R \rangle$ with $A' \subseteq A$.

Suppose: m is the maximum of A'; we need to prove that m is then also a maximal element of A'.

To this end, consider an arbitrary element $x \in A'$ and suppose that $m\ R\ x$. Since m is the maximum of A', it holds that $x\ R\ m$.

Since R is a reflexive ordering, and hence it is antisymmetric, from $x\ R\ m$ and $m\ R\ x$ it follows that $x = m$.

Thereby it is now proved that, for all $x \in A'$, if $m\ R\ x$, then $x = m$, so m is a maximal element of A'. □

Lexicografic ordering (irreflexive)

Let $\langle A_1, R_1\rangle$ and $\langle A_2, R_2\rangle$ be *irreflexive* orderings.
We define R_{lex} on $A_1 \times A_2$ by:

$$(x, y)\ R_{\text{lex}}\ (x', y') \quad \text{if } x\ R_1\ x' \vee (x = x' \wedge y\ R_2\ y')\ .$$

Examples

Take $\langle \mathbb{N}, <\rangle$ and $\langle \mathbb{N}, <\rangle$ for $\langle A_1, R_1\rangle$ and $\langle A_2, R_2\rangle$, and $<_{\text{lex}}$ on $\mathbb{N} \times \mathbb{N}$ for R_{lex}.

- $(5, 8) <_{\text{lex}} (7, 2)$, because $5 < 7$
- $(5, 8) <_{\text{lex}} (5, 11)$, because $5 = 5$ and $8 < 11$
- $(5, 8) <_{\text{lex}} (4, 11)$ does not hold
- $(5, 8) <_{\text{lex}} (5, 2)$ does not hold
- $(5, 8) <_{\text{lex}} (5, 8)$ does not hold