# Propositional Logic

Lectures 1 and 2 (Chapters 2–7)





## Propositions

*A proposition is a statement that is true or false.*

### Examples:

- grass is green
- $5 > 3$
- $5 < 3$
- grass is green and roses are blue
- if $x > 1$, then $x^2 \neq x$

### Non-examples:

- What time is it?
- Don't look back!





## Abstract propositions: vocabulary

*We want to study logic without being distracted by the concrete contents of propositions!*

### Proposition variables:

$a$, $b$, $c$, ...

### Connectives:

- $\wedge$ 'and'
- $\vee$ 'or'
- $\neg$ 'not'
- $\Rightarrow$ 'if ... then ...'
- $\Leftrightarrow$ 'if and only if'

### Examples:

| | |
|---|---|
| grass is green | $a$ |
| $5 > 3$ | $a$ |
| $5 < 3$ | $a$ |
| grass is green and roses are blue | $a \wedge b$ |
| if $x > 1$, then $x^2 \neq x$ | $a \Rightarrow \neg b$ |





## Abstract propositions: grammar

### Inductive definition of abstract propositions:

1. (Basis) every proposition variable is an abstract proposition;
2. (Step)
   - 2.1 if $P$ is an abstract proposition, then so is $(\neg P)$;
   - 2.2 if $P$ and $Q$ are abstract propositions, then so are $(P \wedge Q)$, $(P \vee Q)$, $(P \Rightarrow Q)$, and $(P \Leftrightarrow Q)$.

($\neg$ is *unary*; $\wedge$, $\vee$, $\Rightarrow$, and $\Leftrightarrow$ are *binary*)

### Examples:

$a$
$b$
$(\neg a)$
$((\neg a) \wedge b)$
$(((\neg a) \wedge b) \vee b)$

### Non-examples:

$a\wedge$
$\Rightarrow\Rightarrow a$
$a\neg b$

## Abstract propositions: grammar



### Inductive definition of abstract propositions:

1. (BASIS) every proposition variable is an abstract proposition;
2. (STEP)
   - 2.1 if $P$ is an abstract proposition, then so is $(\neg P)$;
   - 2.2 if $P$ and $Q$ are abstract propositions, then so are $(P \wedge Q)$, $(P \vee Q)$, $(P \Rightarrow Q)$, and $(P \Leftrightarrow Q)$.

($\neg$ is *unary*; $\wedge$, $\vee$, $\Rightarrow$, and $\Leftrightarrow$ are *binary*)

Show that $(((\neg a) \wedge b) \vee b)$ is indeed an abstract proposition:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{a}\,1}{(\neg a)}\,2.1 \quad \dfrac{}{b}\,1}{((\neg a) \wedge b)}\,2.2 \quad \dfrac{}{b}\,1}{(((\neg a) \wedge b) \vee b)}\,2.2$$



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## Omitting parentheses:



*We want to omit as many parentheses from abstract propositions as possible, but without causing ambiguity.*

1. Outermost parentheses can always be omitted
2. We agree on the following priority schema:

| |
|---|
| $\neg$ |
| $\wedge$ $\vee$ |
| $\Rightarrow$ |
| $\Leftrightarrow$ |

NB: since $\neg$ has highest priority, parentheses around a negation may always be omitted. We may, e.g., also omit the parentheses from $\neg(\neg a)$.

### Examples:

| | | | | | |
|---|---|---|---|---|---|
| $((\neg a) \vee (\neg b))$ | $\overset{1}{\rightsquigarrow}$ | $(\neg a) \vee (\neg b)$ | $\overset{2}{\rightsquigarrow}$ | $\neg a \vee \neg b$ | |
| $((\neg a) \wedge b)$ | $\overset{1}{\rightsquigarrow}$ | $(\neg a) \wedge b$ | $\overset{2}{\rightsquigarrow}$ | $\neg a \wedge b$ | |
| $(\neg(a \wedge b))$ | $\overset{1}{\rightsquigarrow}$ | $\neg(a \wedge b)$ | ~~$\overset{?}{\rightsquigarrow}$~~ | ~~$\neg a \wedge b$~~ | NO! |



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## Negation



A negation $\neg P$ is

*true if $P$ is false;*
*false if $P$ is true.*

### Truth table for $\neg$:

| $P$ | $\neg P$ |
|---|---|
| 0 | 1 |
| 1 | 0 |

0 = false
1 = true



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## Conjunction



A conjunction $P \wedge Q$ is

*true if $P$ is true and $Q$ is true;*
*false in all other cases.*

### Truth table for $\wedge$:

| $P$ | $Q$ | $P \wedge Q$ |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 0 |
| 1 | 0 | 0 |
| 1 | 1 | 1 |

We first list all possible combinations of assignments to $P$ and $Q$.



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# Disjunction



A disjunction $P \vee Q$ is

*true if $P$ is true, or $Q$ is true, or both;*
*false otherwise.*

## Truth table for $\vee$:

| $P$ | $Q$ | $P \vee Q$ |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 1 |
| 1 | 0 | 1 |
| 1 | 1 | 1 |





# Inclusive vs. Exclusive Or



## Examples:

- Inclusive:
  *Can you show me a passport or drivers license?*
- Exclusive:
  *Do you want peanut butter or jam on your sandwich?*

## Exercise:

Give abstract proposition that 'behaves' as exclusive or of $a$ and $b$ (notation: $a \oplus b$).





# Implication



An implication $P \Rightarrow Q$ is

*true if whenever $P$ is true, then also $Q$ is true;*
*false otherwise.*

## Truth table for $\Rightarrow$:

| $P$ | $Q$ | $P \Rightarrow Q$ |
|---|---|---|
| 0 | 0 | 1 |
| 0 | 1 | 1 |
| 1 | 0 | 0 |
| 1 | 1 | 1 |

When is $P \Rightarrow Q$ clearly not true?
Now consider $n > 2 \Rightarrow n + 1 > 2$.
Clearly, this implication is true for every $n$.

$n = 1$: both $n > 2$ and $n + 1 > 2$ false.
$n = 2$: $n > 2$ false, $n + 1 > 2$ true.
$n = 3$: both $n > 2$ and $n + 1 > 2$ true.





# Biimplication



A biimplication $P \Leftrightarrow Q$ is

*true if $P$ and $Q$ have the same truth value;*
*false otherwise.*

## Truth table for $\Leftrightarrow$:

| $P$ | $Q$ | $P \Leftrightarrow Q$ |
|---|---|---|
| 0 | 0 | 1 |
| 0 | 1 | 0 |
| 1 | 0 | 0 |
| 1 | 1 | 1 |

# Computing truth table



Example:

Computing the truth table of $\neg(a \Rightarrow b)$:

| $a$ | $b$ | $a \Rightarrow b$ | $\neg(a \Rightarrow b)$ |
|---|---|---|---|
| 0 | 0 | 1 | 0 |
| 0 | 1 | 1 | 0 |
| 1 | 0 | 0 | 1 |
| 1 | 1 | 1 | 0 |





# Tautology



*An abstract proposition is a tautology if its column in a truth table exclusively consists of* $1$*s.*

Examples:

- $a \vee \neg a$
- $a \Rightarrow a$
- $a \Rightarrow (b \Rightarrow a)$
- ...

| $a$ | $b$ | $b \Rightarrow a$ | $a \Rightarrow (b \Rightarrow a)$ |
|---|---|---|---|
| 0 | 0 | 1 | 1 |
| 0 | 1 | 0 | 1 |
| 1 | 0 | 1 | 1 |
| 1 | 1 | 1 | 1 |





# Contradiction



*An abstract proposition is a contradiction if its column in a truth table exclusively consists of* $0$*s.*

Examples:

- $a \wedge \neg a$
- $(a \Rightarrow b) \wedge (a \wedge \neg b)$
- ...

| $a$ | $b$ | $a \Rightarrow b$ | $\neg b$ | $a \wedge \neg b$ | $(a \Rightarrow b) \wedge (a \wedge \neg b)$ |
|---|---|---|---|---|---|
| 0 | 0 | 1 | 1 | 0 | 0 |
| 0 | 1 | 1 | 0 | 0 | 0 |
| 1 | 0 | 0 | 1 | 1 | 0 |
| 1 | 1 | 1 | 0 | 0 | 0 |





# Contingency



*An abstract proposition is contingent if it is not a tautology, nor a contradiction.*

Examples:

1. $a$
2. $a \Rightarrow \neg a$
3. $a \wedge b$
4. $a \vee b$
5. $\neg a \Rightarrow (b \wedge c)$

# Equivalence



## Example:

Combine truth tables for $\neg(a \Rightarrow b)$ and $\neg(\neg a \vee b)$:

| $a$ | $b$ | $a \Rightarrow b$ | $\neg(a \Rightarrow b)$ | $\neg a$ | $\neg a \vee b$ | $\neg(\neg a \vee b)$ |
|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 0 | 1 | 1 | 0 |
| 0 | 1 | 1 | 0 | 1 | 1 | 0 |
| 1 | 0 | 0 | 1 | 0 | 0 | 1 |
| 1 | 1 | 1 | 0 | 0 | 1 | 0 |

Note: the columns for $\neg(a \Rightarrow b)$ and $\neg(\neg a \vee b)$ are identical.

*Abstract propositions with identical columns in a combined truth table are said to be equivalent.*



# Equivalence



All tautologies are equivalent.
We introduce an extra symbol `True` to denote an arbitrary tautology.

All contradictions are equivalent.
We introduce an extra symbol `False` to denote an arbitrary contradiction.

Not all contingencies are equivalent.



# Abstract propositions: grammar (extended)



From now on, we shall consider `True` and `False` officially as part of the vocabulary of abstract propositions.

## Inductive definition of abstract propositions:

1. (BASIS) `True` and `False` are abstract propositions, and every proposition variable is an abstract proposition;
2. (STEP)
   2.1 if $P$ is an abstract proposition, then so is $(\neg P)$;
   2.2 if $P$ and $Q$ are abstract propositions, then so are $(P \wedge Q)$, $(P \vee Q)$, $(P \Rightarrow Q)$, and $(P \Leftrightarrow Q)$.

($\neg$ is *unary*; $\wedge$, $\vee$, $\Rightarrow$, and $\Leftrightarrow$ are *binary*)



# Notation



If $P$ is equivalent to $Q$, then we write $P \stackrel{val}{==} Q$.

Note: $\stackrel{val}{==}$ is not part of the vocabulary of the language of abstract propositions; it is a *meta-symbol*.

So:

$$\underbrace{\underbrace{a \Rightarrow b}_{\text{abstr. prop.}} \stackrel{val}{==} \underbrace{\neg a \vee b}_{\text{abstr. prop.}}}_{\text{meta-formula}}$$

## Proving equivalences



Suppose we want to prove that $\neg(Q \Rightarrow R) \stackrel{val}{==} \neg R \wedge Q$.

How did we proceed (until now)?

We would prove the equivalence using truth tables:

| $Q$ | $R$ | $Q \Rightarrow R$ | $\neg(Q \Rightarrow R)$ | $\neg R$ | $\neg R \wedge Q$ |
|---|---|---|---|---|---|
| 0 | 0 | 1 | 0 | 1 | 0 |
| 0 | 1 | 1 | 0 | 0 | 0 |
| 1 | 0 | 0 | 1 | 1 | 1 |
| 1 | 1 | 1 | 0 | 0 | 0 |

The columns of $\neg(Q \Rightarrow R)$ and $\neg R \wedge Q$ are identical, so it holds that $\neg(Q \Rightarrow R) \stackrel{val}{==} \neg R \wedge Q$.



## Commutativity, Associativity



**Commutativity:**

$P \wedge Q \stackrel{val}{==} Q \wedge P$

$P \vee Q \stackrel{val}{==} Q \vee P$

$P \Leftrightarrow Q \stackrel{val}{==} Q \Leftrightarrow P$

**Associativity:**

$(P \wedge Q) \wedge R \stackrel{val}{==} P \wedge (Q \wedge R)$

$(P \vee Q) \vee R \stackrel{val}{==} P \vee (Q \vee R)$

$(P \Leftrightarrow Q) \Leftrightarrow R \stackrel{val}{==} P \Leftrightarrow (Q \Leftrightarrow R)$

NB: $P \Rightarrow Q \stackrel{val}{\neq} Q \Rightarrow P$

NB: $P \Rightarrow (Q \Rightarrow R) \stackrel{val}{\neq} (P \Rightarrow Q) \Rightarrow R$

| $P$ | $Q$ | $P \Rightarrow Q$ | $Q \Rightarrow P$ |
|---|---|---|---|
| 0 | 1 | 1 | 0 |

| $P$ | $Q$ | $R$ | $P \Rightarrow (Q \Rightarrow R)$ | $(P \Rightarrow Q) \Rightarrow R$ |
|---|---|---|---|---|
| 0 | 1 | 0 | 1 | 0 |

NB: In view of Associativity, we shall write $P \wedge Q \wedge R$ to denote both $(P \wedge Q) \wedge R$ and $P \wedge (Q \wedge R)$.



## Be Careful!



Are the abstract propositions

$(a \wedge b) \vee c$

and

$a \wedge (b \vee c)$

equivalent?

NO! (see p. 19 in your book)

This shows that the (remaining) parentheses in these abstract propositions are important. In fact, they make the difference!



## Idempotence, Double negation



**Idempotence:**

$P \wedge P \stackrel{val}{==} P$

$P \vee P \stackrel{val}{==} P$

NB: $P \Rightarrow P \stackrel{val}{\neq} P$

$P \Leftrightarrow P \stackrel{val}{\neq} P$

**Double Negation:**

$\neg\neg P \stackrel{val}{==} P$

‘It is **not** that I **don’t** like spinach’

(NB: in propositional logic the intended nuance cannot be captured.)

## True and False



**Inversion:**

$\neg\mathtt{True} \stackrel{val}{==} \mathtt{False}$

$\neg\mathtt{False} \stackrel{val}{==} \mathtt{True}$

**Negation:**

$\neg P \stackrel{val}{==} P \Rightarrow \mathtt{False}$

**Contradiction:**

$P \wedge \neg P \stackrel{val}{==} \mathtt{False}$

**Excluded Middle:**

$P \vee \neg P \stackrel{val}{==} \mathtt{True}$

True/False**-elimination:**

$P \wedge \mathtt{True} \stackrel{val}{==} P$

$P \wedge \mathtt{False} \stackrel{val}{==} \mathtt{False}$

$P \vee \mathtt{True} \stackrel{val}{==} \mathtt{True}$

$P \vee \mathtt{False} \stackrel{val}{==} P$



## Distributivity and De Morgan



**Distributivity:**

$P \wedge (Q \vee R) \stackrel{val}{==} (P \wedge Q) \vee (P \wedge R)$

$P \vee (Q \wedge R) \stackrel{val}{==} (P \vee Q) \wedge (P \vee R)$

**De Morgan:**

$\neg(P \wedge Q) \stackrel{val}{==} \neg P \vee \neg Q$

$\neg(P \vee Q) \stackrel{val}{==} \neg P \wedge \neg Q$



## Implication and Contraposition



**Implication:**

$P \Rightarrow Q \stackrel{val}{==} \neg P \vee Q$

$$
\begin{array}{ll}
 & \neg P \Rightarrow Q \\
\stackrel{val}{==} & \{\ \text{Implication}\ \} \\
 & \neg\neg P \vee Q \\
\stackrel{val}{==} & \{\ \text{Double Negation}\ \} \\
 & P \vee Q
\end{array}
$$

**Contraposition:**

$P \Rightarrow Q \stackrel{val}{==} \neg Q \Rightarrow \neg P$

NB: $P \Rightarrow Q \stackrel{val}{\neq} \neg P \Rightarrow \neg Q$

(NB: 'if . . . , then . . . ' in natural language often means 'if and only if'.)



## Bi-implication and self-equivalence



**Bi-implication:**

$P \Leftrightarrow Q \stackrel{val}{==} (P \Rightarrow Q) \wedge (Q \Rightarrow P)$

**Self-equivalence:**

$P \Leftrightarrow P \stackrel{val}{==} \mathtt{True}$

# Table of standard equivalences



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**Equivalences for connectives**

| | |
|---|---|
| **Commutativity:**<br>$P \wedge Q \stackrel{val}{=} Q \wedge P$,<br>$P \vee Q \stackrel{val}{=} Q \vee P$,<br>$P \Leftrightarrow Q \stackrel{val}{=} Q \Leftrightarrow P$ | **Associativity:**<br>$(P \wedge Q) \wedge R \stackrel{val}{=} P \wedge (Q \wedge R)$,<br>$(P \vee Q) \vee R \stackrel{val}{=} P \vee (Q \vee R)$,<br>$(P \Leftrightarrow Q) \Leftrightarrow R \stackrel{val}{=} P \Leftrightarrow (Q \Leftrightarrow R)$ |
| **Idempotence:**<br>$P \wedge P \stackrel{val}{=} P$,<br>$P \vee P \stackrel{val}{=} P$ | **Double Negation:**<br>$\neg\neg P \stackrel{val}{=} P$ |
| **Inversion:**<br>$\neg\texttt{True} \stackrel{val}{=} \texttt{False}$,<br>$\neg\texttt{False} \stackrel{val}{=} \texttt{True}$ | **True/False-elimination:**<br>$P \wedge \texttt{True} \stackrel{val}{=} P$,<br>$P \wedge \texttt{False} \stackrel{val}{=} \texttt{False}$,<br>$P \vee \texttt{True} \stackrel{val}{=} \texttt{True}$,<br>$P \vee \texttt{False} \stackrel{val}{=} P$ |
| **Negation:**<br>$\neg P \stackrel{val}{=} P \Rightarrow \texttt{False}$ | **Contradiction:**<br>$P \wedge \neg P \stackrel{val}{=} \texttt{False}$<br>**Excluded Middle:**<br>$P \vee \neg P \stackrel{val}{=} \texttt{True}$ |
| **Distributivity:**<br>$P \wedge (Q \vee R) \stackrel{val}{=} (P \wedge Q) \vee (P \wedge R)$,<br>$P \vee (Q \wedge R) \stackrel{val}{=} (P \vee Q) \wedge (P \vee R)$ | **De Morgan:**<br>$\neg(P \wedge Q) \stackrel{val}{=} \neg P \vee \neg Q$,<br>$\neg(P \vee Q) \stackrel{val}{=} \neg P \wedge \neg Q$ |
| **Implication:**<br>$P \Rightarrow Q \stackrel{val}{=} \neg P \vee Q$ | **Contraposition:**<br>$P \Rightarrow Q \stackrel{val}{=} \neg Q \Rightarrow \neg P$ |
| **Bi-implication:**<br>$P \Leftrightarrow Q \stackrel{val}{=} (P \Rightarrow Q) \wedge (Q \Rightarrow P)$ | **Self-equivalence:**<br>$P \Leftrightarrow P \stackrel{val}{=} \texttt{True}$ |

For the collection of *all* standard equivalences, see page 372 of the book!

You will have to know them by heart (including their names!).

Start memorising them today!



# Calculation



Recall the following *calculation*:

$$\begin{array}{ll} & \neg P \Rightarrow Q \\ \stackrel{val}{=} & \{ \text{ Implication } \} \\ & \neg\neg P \vee Q \\ \stackrel{val}{=} & \{ \text{ Double Negation } \} \\ & P \vee Q \end{array}$$

Can we conclude

$$\neg P \Rightarrow Q \stackrel{val}{=} P \vee Q \ ?$$

1. What about applying two standard equivalences in a row?
   Does it preserve equivalence?
2. First step: not a *literal* application of Implication.
   Can we do substitutions?
3. Second step: literal application of Double Negation.
   Is it safe to apply standard equivalences in a larger context?



# $\stackrel{val}{=}$ is a decent *equivalence*



## Lemma 6.1.1

1. (Reflexivity:) $P \stackrel{val}{=} P$
2. (Symmetry:) If $P \stackrel{val}{=} Q$, then $Q \stackrel{val}{=} P$
3. (Transitivity:) If $P \stackrel{val}{=} Q$ and $Q \stackrel{val}{=} R$, then $P \stackrel{val}{=} R$



# Substitution



*Substitution is the replacement of all occurrences of a 'letter' by a formula.*

## Examples:

1. If we substitute $Q \wedge P$ for $P$ in the valid equivalence

$$P \Rightarrow Q \stackrel{val}{=} \neg P \vee Q \ ,$$

then we get the valid equivalence

$$(Q \wedge P) \Rightarrow Q \stackrel{val}{=} \neg(Q \wedge P) \vee Q \ .$$

## Substitution



*Substitution is the replacement of all occurrences of a 'letter' by a formula.*

Examples:

2. If we substitute $\neg R$ for $Q$ in the valid equivalence

$$(Q \wedge P) \Rightarrow Q \stackrel{val}{==} \neg(Q \wedge P) \vee Q \ ,$$

then we get the valid equivalence

$$(\neg R \wedge P) \Rightarrow \neg R \stackrel{val}{==} \neg(\neg R \wedge P) \vee \neg R \ .$$



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## Substitution



*Substitution is the replacement of all occurrences of a 'letter' by a formula.*

Examples:

3. If we (simultaneously) substitute $Q \wedge P$ for $P$ and $\neg R$ for $Q$ in the valid equivalence

$$P \Rightarrow Q \stackrel{val}{==} \neg P \vee Q \ ,$$

then we get the valid equivalence

$$(Q \wedge P) \Rightarrow \neg R \stackrel{val}{==} \neg(Q \wedge P) \vee \neg R \ .$$



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## Substitution Rule



SUBSTITUTION PRESERVES EQUIVALENCE

Important remarks:

1. Substitution operates on entire equivalences
2. If you substitute for some letter $P$ in an equivalence, then you have to replace all occurrences of $P$ in that equivalence!



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## Leibniz's rule



*Leibniz's rule is about the replacement of a subformula by an equivalent subformula.*

Example:

From the valid equivalence

$$P \Rightarrow Q \stackrel{val}{==} \neg P \vee Q$$

we can *make new valid equivalences* by replacing $P \Rightarrow Q$ in some complex formula by $\neg P \vee Q$, for instance:

$$(\neg P \wedge (P \Rightarrow Q)) \vee R \stackrel{val}{==} (\neg P \wedge (\neg P \vee Q)) \vee R$$



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# Leibniz's rule



*Leibniz's rule is about the replacement of a subformula by an equivalent subformula.*

Schematically:

$$\frac{P \overset{val}{=\!=} Q}{\cdots P \cdots \overset{val}{=\!=} \cdots Q \cdots}$$



# Proving tautologies—method 1



To *prove with a calculation* that $P$ is a tautology:

Give calculation that shows $P \overset{val}{=\!=} \texttt{True}$.



# Proving tautologies—method 1 (example)



Prove with a calculation that $\neg(P \wedge \neg P)$ is a tautology.

We have the following calculation:

$$\begin{array}{ll} & \neg(P \wedge \neg P) \\ \overset{val}{=\!=} & \{ \text{ De Morgan } \} \\ & \neg P \vee \neg\neg P \\ \overset{val}{=\!=} & \{ \text{ Double Negation } \} \\ & P \vee \neg P \\ \overset{val}{=\!=} & \{ \text{ Excluded Middle } \} \\ & \texttt{True} \end{array}$$

So $\neg(P \wedge \neg P)$ is a tautology.



# Proving tautologies—method 2



Lemma 6.1.3

If $P \overset{val}{=\!=} Q$, then $P \Leftrightarrow Q$ is a tautology, and vice versa.

To *prove with a calculation* that $P$ is a tautology:

Give a calculation that shows $P \overset{val}{=\!=} Q$.

## Proving tautologies—method 2 (example)



Prove with a calculation that $\neg(Q \Rightarrow R) \Leftrightarrow (\neg R \wedge Q)$ is a tautology.

First, we establish, with a calculation, that $\neg(Q \Rightarrow R) \stackrel{val}{==} (\neg R \wedge Q)$:

$\neg(Q \Rightarrow R)$
$\stackrel{val}{==}$ { Implication }
$\neg(\neg Q \vee R)$
$\stackrel{val}{==}$ { De Morgan }
$\neg\neg Q \wedge \neg R$
$\stackrel{val}{==}$ { Double negation }
$\neg R \wedge Q$

Explanation:

Substituting $Q$ for $P$ and $R$ for $Q$ in
$P \Rightarrow Q \stackrel{val}{==} \neg P \vee Q$ (Implication)
we get, by the substitution rule:
$Q \Rightarrow R \stackrel{val}{==} \neg Q \vee R$.

(The application of this equivalence in the calculation involves an application of Leibniz.)



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## Proving tautologies—method 2 (example)



Prove with a calculation that $\neg(Q \Rightarrow R) \Leftrightarrow (\neg R \wedge Q)$ is a tautology.

First, we establish, with a calculation, that $\neg(Q \Rightarrow R) \stackrel{val}{==} (\neg R \wedge Q)$:

$\neg(Q \Rightarrow R)$
$\stackrel{val}{==}$ { Implication }
$\neg(\neg Q \vee R)$
$\stackrel{val}{==}$ { De Morgan }
$\neg\neg Q \wedge \neg R$
$\stackrel{val}{==}$ { Double negation }
$\neg R \wedge Q$

Explanation:

Substituting $\neg Q$ for $P$ and $R$ for $Q$ in
$\neg(P \vee Q) \stackrel{val}{==} \neg P \wedge \neg Q$
(De Morgan)
we get, by the substitution rule:
$\neg(\neg Q \vee R) \stackrel{val}{==} \neg\neg Q \wedge \neg R$.



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## Proving tautologies—method 2 (example)



Prove with a calculation that $\neg(Q \Rightarrow R) \Leftrightarrow (\neg R \wedge Q)$ is a tautology.

First, we establish, with a calculation, that $\neg(Q \Rightarrow R) \stackrel{val}{==} (\neg R \wedge Q)$:

$\neg(Q \Rightarrow R)$
$\stackrel{val}{==}$ { Implication }
$\neg(\neg Q \vee R)$
$\stackrel{val}{==}$ { De Morgan }
$\neg\neg Q \wedge \neg R$
$\stackrel{val}{==}$ { Double negation }
$\neg R \wedge Q$

Explanation:

Substituting $Q$ for $P$ in
$\neg\neg P \stackrel{val}{==} P$ (Double negation)
we get, by the substitution rule:
$\neg\neg Q \stackrel{val}{==} Q$.

(The application of this equivalence in the calculation involves an application of Leibniz, and is followed by an application of Commutativity.)



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## Proving tautologies—method 2 (example)



Prove with a calculation that $\neg(Q \Rightarrow R) \Leftrightarrow (\neg R \wedge Q)$ is a tautology.

First, we establish, with a calculation, that $\neg(Q \Rightarrow R) \stackrel{val}{==} (\neg R \wedge Q)$:

$\neg(Q \Rightarrow R)$
$\stackrel{val}{==}$ { Implication }
$\neg(\neg Q \vee R)$
$\stackrel{val}{==}$ { De Morgan }
$\neg\neg Q \wedge \neg R$
$\stackrel{val}{==}$ { Double negation }
$\neg R \wedge Q$

From $\neg(Q \Rightarrow R) \stackrel{val}{==} \neg R \wedge Q$ it follows (by Lemma 6.1.3) that $\neg(Q \Rightarrow R) \Leftrightarrow (\neg R \wedge Q)$ is a tautology.



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## Logical Consequence



Recall:

$P \stackrel{val}{==} Q$ means $\begin{cases} \text{(a) whenever } P \text{ is } 1, \text{ then also } Q \text{ is } 1 \\ \text{(b) whenever } Q \text{ is } 1, \text{ then also } P \text{ is } 1 \end{cases}$

Define:

$P \stackrel{val}{\models} Q$ means { (a) whenever $P$ is $1$, then also $Q$ is $1$

Pronounce $P \stackrel{val}{\models} Q$ as "*$P$ is stronger than $Q$*."

| $P$ | $Q$ |
|---|---|
| 1 → | 1 |
| 0 | 1/0 |

$P \stackrel{val}{\models} Q$: 1s are *carried over* from $P$ to $Q$.



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## Logical Consequence (example 1)



$\neg P \left\{ \begin{matrix} \stackrel{val}{\models} ? \\ \stackrel{val}{=\!\!\!|} ? \end{matrix} \right\} P \Rightarrow Q$

| $P$ | $Q$ | $\neg P$ | $P \Rightarrow Q$ |
|---|---|---|---|
| 0 | 0 | 1 → | 1 |
| 0 | 1 | 1 → | 1 |
| 1 | 0 | 0 | 0 |
| 1 | 1 | 0 | 1 (extra true) |

extra true

So $\neg P$ is stronger than $P \Rightarrow Q$ (i.e., $\neg P \stackrel{val}{\models} P \Rightarrow Q$).



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## Logical Consequence (example 2)



$P \Rightarrow Q \left\{ \begin{matrix} \stackrel{val}{\models} ? \\ \stackrel{val}{=\!\!\!|} ? \end{matrix} \right\} P \vee Q$

| $P$ | $Q$ | $P \Rightarrow Q$ | $P \vee Q$ |
|---|---|---|---|
| 0 | 0 | 1 →✗ | 0 |
| 0 | 1 | 1 | 1 |
| 1 | 0 | 0 | ✗← 1 |
| 1 | 1 | 1 | 1 |

So $P \Rightarrow Q$ and $P \vee Q$ are incomparable.



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## Standard weakenings



**∧-∨-weakening:**

$P \wedge Q \stackrel{val}{\models} P$

$P \stackrel{val}{\models} P \vee Q$

| $P$ | $Q$ | $P \wedge Q$ | $P$ | $P \vee Q$ |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 0 | 1 | 0 | 0 | 1 |
| 1 | 0 | 0 | 1 → | 1 |
| 1 | 1 | 1 → | 1 → | 1 |

Also:

$P \wedge Q \stackrel{val}{\models} Q$ and

$Q \stackrel{val}{\models} P \vee Q$ .

**Extremes:**

`False` $\stackrel{val}{\models} P$

$P \stackrel{val}{\models}$ `True`

`False` is strongest of all

`True` is weakest of all



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## Basic properties of $\stackrel{val}{\models}$ , $\stackrel{val}{=\!\!|}$



Lemma 7.3.1

(1a) $P \stackrel{val}{\models} P$.

(2) If $P \stackrel{val}{\models} Q$, then $Q \stackrel{val}{=\!\!|} P$, and vice versa.

(3) If $P \stackrel{val}{\models} Q$ and $Q \stackrel{val}{\models} R$, then $P \stackrel{val}{\models} R$.



## Basic properties of $\stackrel{val}{\models}$ , $\stackrel{val}{=\!\!|}$



Lemma 7.3.2

$P \stackrel{val}{==} Q$ if, and only if, $P \stackrel{val}{\models} Q$ and $P \stackrel{val}{=\!\!|} Q$.

*So, if you need to prove $P \stackrel{val}{\models} Q$ or $P \stackrel{val}{=\!\!|} Q$ by a calculation, then it is enough to prove $P \stackrel{val}{==} Q$. But $P \stackrel{val}{\models} Q$ (or $P \stackrel{val}{=\!\!|} Q$) alone is not enough to conclude $P \stackrel{val}{==} Q$!*



## Basic properties of $\stackrel{val}{\models}$ , $\stackrel{val}{=\!\!|}$



Lemma 7.3.4

$P \stackrel{val}{\models} Q$ if, and only if, $P \Rightarrow Q$ is a tautology.



## Substitution Rule for $\stackrel{val}{\models}$ , $\stackrel{val}{=\!\!|}$



The Substitution Rule also works for $\stackrel{val}{\models}$ and $\stackrel{val}{=\!\!|}$ :

| SUBSTITUTION PRESERVES WEAKENING/STRENGHTENING |
|---|

Example

We have the following valid weakening:

$$P \wedge Q \stackrel{val}{\models} P \vee R$$

and hence, according to the Substitution Rule, if we substitute $(Q \Rightarrow R)$ for $P$ and $(P \vee Q)$ for $Q$, we get another valid weakening:

$$(Q \Rightarrow R) \wedge (P \vee Q) \stackrel{val}{\models} (Q \Rightarrow R) \vee R \ .$$

# Leibniz for $\stackrel{val}{\models}$, $\stackrel{val}{\models}$ ?



Recall Leibniz's Rule for making new equivalences:

$$\frac{P \stackrel{val}{=} Q}{\ldots P \ldots \stackrel{val}{=} \ldots Q \ldots}$$

Can we replace $\stackrel{val}{=}$ by $\stackrel{val}{\models}$ in this rule?

## Examples

Note that, by $\wedge$-$\vee$-weakening, $P \wedge Q \stackrel{val}{\models} P \vee Q$. Now consider:

1. $\neg(P \wedge Q) \stackrel{val}{\not\models} \neg(P \vee Q)$;
2. $R \Rightarrow (P \wedge Q) \stackrel{val}{\models} R \Rightarrow (P \vee Q)$; and
3. $(P \wedge Q) \Rightarrow R \stackrel{val}{\not\models} (P \vee Q) \Rightarrow R$.

Conclusion: replacing $\stackrel{val}{=}$ by $\stackrel{val}{\models}$ does not yield a valid rule!



# Monotonicity



We do have the following weaker variant of Leibniz's Rule:

**Monotonicity:**
(1) If $P \stackrel{val}{\models} Q$, then $P \wedge R \stackrel{val}{\models} Q \wedge R$
(2) If $P \stackrel{val}{\models} Q$, then $P \vee R \stackrel{val}{\models} Q \vee R$

## Example:

Since $P \stackrel{val}{\models} P \vee Q$ by $\wedge$-$\vee$-weakening, we have:

$P \wedge R$

$\stackrel{val}{\models}$ { $\wedge$-$\vee$-weakening + Monotonicity }

$(P \vee Q) \wedge R$ .



# Example



Prove with a calculation that $\neg(P \Rightarrow Q) \Rightarrow (\neg Q \wedge (P \vee R))$ is a tautology.

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First, we establish that $\neg(P \Rightarrow Q) \stackrel{val}{\models} \neg Q \wedge (P \vee R)$:

$\neg(P \Rightarrow Q)$

$\stackrel{val}{=}$ { Implication }

$\neg(\neg P \vee Q)$

$\stackrel{val}{=}$ { De Morgan }

$\neg\neg P \wedge \neg Q$

$\stackrel{val}{=}$ { Double Negation }

$P \wedge \neg Q$

$\stackrel{val}{\models}$ { $\wedge$-$\vee$-weakening + Monotonicity }

$\neg Q \wedge (P \vee R)$

So, by Lemma 7.3.4, the formula

$$\neg(P \Rightarrow Q) \Rightarrow (\neg Q \wedge (P \vee R))$$

is a tautology.



# Formal System for Calculation



We now have a precisely defined formal system for calculating with abstract propositions:

- **standard equivalences** and **standard weakenings**;
- **inference rules** (viz. reflexivity, symmetry, transitivity, substitution, Leibniz for equality, Monotonicity for weakening)

It gives a method to prove in a structured manner that

- two abstract propositions are equivalent, or one is stronger/weaker than the other;
- an abstract proposition is a tautology.