Reasoning

Lecture 4, 5 and 6 (Chapters 11–15)

Calculation: strenghts and weaknesses

$$
\begin{aligned}
& P \wedge (P \vee Q) \\
\stackrel{val}{=\!=} \; & \{ \texttt{True/False}\text{-elimination} \} \\
& (P \vee \texttt{False}) \wedge (P \vee Q) \\
\stackrel{val}{=\!=} \; & \{ \text{Distributivity} \} \\
& P \vee (\texttt{False} \wedge Q) \\
\stackrel{val}{=\!=} \; & \{ \texttt{True/False}\text{-elimination } (2\times) \} \\
& P
\end{aligned}
$$

Conclusions:

- $P \wedge (P \vee Q) \stackrel{val}{=\!=} P$
- $P \wedge (P \vee Q) \Leftrightarrow P$ is a tautology

Calculation gives a precisely defined formal system for showing equivalence or weakening.

It can also be used to prove tautologies, but the formal system is not designed especially for this purpose.

An example of a mathematical proof

Theorem
For all $x \in \mathbb{Z}$, if x is even, then so is x^2.

Proof:
Let $x \in \mathbb{Z}$, and suppose x is even;
we need to prove that x^2 is even too.

If x is even, then $x = 2y$ for some $y \in \mathbb{Z}$.
Then $x^2 = (2y)^2 = 4y^2 = 2(2y^2)$ and $2y^2 \in \mathbb{Z}$.

So x^2 is even.

(sub)goal
generating hypothesis
pure hypothesis
conclusion

Exposing logical structure

Let $x \in \mathbb{Z}$

Suppose: x is even

$x = 2y$ for some $y \in \mathbb{Z}$

$x^2 = (2y)^2 = 4y^2 = 2(2y^2)$

$2y^2 \in \mathbb{Z}$

x^2 is even

For all $x \in \mathbb{Z}$, if x is even, then so is x^2

(sub)goal
generating hypothesis
pure hypothesis
conclusion

Inference

Q is a correct conclusion from n premisses $P_1, P_2, \ldots, P_n$
if, and only if,
$P_1 \wedge P_2 \wedge \cdots \wedge P_n \stackrel{val}{\models} Q$

If $n = 0$, then $P_1 \wedge P_2 \wedge \cdots \wedge P_n = \texttt{True}$.

Note: if $\texttt{True} \stackrel{val}{\models} Q$, then $Q \stackrel{val}{=} \texttt{True}$!

That is, Q holds unconditionally.

Inference (example)

	premisses	correct conclusion	reason
0		$P \vee \neg P$	$\texttt{True} \stackrel{val}{\models} P \vee \neg P$
1	$\neg P \wedge \neg Q$	$\neg Q$ $\neg(P \vee Q)$	$\neg P \wedge \neg Q \stackrel{val}{\models} \neg Q$ $\neg P \wedge \neg Q \stackrel{val}{\models} \neg(P \vee Q)$
2	$P \Rightarrow Q$ P	Q	$(P \Rightarrow Q) \wedge P \stackrel{val}{\models} Q$
3	$P \vee Q$ $P \Rightarrow R$ $Q \Rightarrow R$	R	$(P \vee Q) \wedge (P \Rightarrow R) \wedge (Q \Rightarrow R) \stackrel{val}{\models} R$

Derivation

The formal system of derivation that we are going to discuss tries to follow more closely the way we usually conduct logical reasonings.

It consists of two types of inference rules:

- elimination rules to draw correct logical conclusions from facts or hypotheses;
- introduction rules to simplify goals.

Formalises the following general proof strategy:

1. using the introduction rules: logically simplify the goal as much as possible (this will typically generate hypotheses);
2. using the elimination rules: combine hypotheses and already established facts to prove simplified goal.

Elimination of conjunction

How can we use a conjunction in a reasoning?

$$P \wedge Q \stackrel{val}{\models} \begin{cases} P \\ Q \end{cases}$$

$\wedge$-elimination:

$\vdots$	
(k)	$P \wedge Q$
$\vdots$	
	{ $\wedge$-elim on (k): }
(ℓ)	P

$(\downarrow)$ $(k < \ell)$

$\vdots$	
(k)	$P \wedge Q$
$\vdots$	
	{ $\wedge$-elim on (k): }
(ℓ)	Q

$(\downarrow)$ $(k < \ell)$

Elimination of implication

How can we use an implication in a reasoning?

$P \Rightarrow Q \overset{val}{\models} ??$

$(P \Rightarrow Q) \wedge P \overset{val}{\models} Q$

$\Rightarrow$-elimination:

	$\vdots$
(k)	$P \Rightarrow Q$
	$\vdots$
(ℓ)	P
	$\vdots$
	{ $\Rightarrow$-elim on (k) and (ℓ): }
(m)	Q

$(\downarrow)$ $(k < m), (\ell < m)$

Introduction of conjunction

How do we establish a conjunction in a reasoning?

$\wedge$-introduction:

	$\vdots$
(k)	P
	$\vdots$
(ℓ)	Q
	{ $\wedge$-intro on (k) and (ℓ): }
(m)	$P \wedge Q$

$(\uparrow)$ $(k < m), (\ell < m)$

Introduction of implication

How do we establish an implication in a reasoning?

$\Rightarrow$-introduction:

	{ Assume: }
(k)	P
	$\vdots$
$(\ell - 1)$	Q
	{ $\Rightarrow$-intro on (k) and $(\ell - 1)$: }
(ℓ)	$P \Rightarrow Q$

$(\uparrow)$

Example

Show with a *derivation* (according to the methods of part II of the book) that the formula

$$(P \Rightarrow (Q \wedge R)) \Rightarrow ((P \Rightarrow Q) \wedge (P \Rightarrow R))$$

is a tautology.

[Derivation on blackboard]

Conjunction and Implication

Derivation rules for $\wedge$ and $\Rightarrow$

$\wedge$-introduction:

$$
\begin{array}{ll}
 & \vdots \\
(k) & P \\
 & \vdots \\
(l) & Q \\
 & \vdots \\
 & \{\ \wedge\text{-intro on } (k) \text{ and } (l):\ \} \\
(m) & P \wedge Q
\end{array}
$$

$\wedge$-elimination:

$$
\begin{array}{ll}
 & ||| \\
(k) & P \wedge Q \\
 & ||| \\
 & \{\ \wedge\text{-elim on } (k):\ \} \\
(m) & P \ \text{resp.}\ Q
\end{array}
$$

$\Rightarrow$-introduction:

$$
\begin{array}{ll}
 & \{\ \text{Assume:}\ \} \\
(k) & \boxed{P} \\
 & \vdots \\
(m-1) & Q \\
 & \{\ \Rightarrow\text{-intro on } (k) \text{ and } (m-1):\ \} \\
(m) & P \Rightarrow Q
\end{array}
$$

$\Rightarrow$-elimination:

$$
\begin{array}{ll}
 & ||| \\
(k) & P \Rightarrow Q \\
 & ||| \\
(l) & P \\
 & ||| \\
 & \{\ \Rightarrow\text{-elim on } (k) \text{ and } (l):\ \} \\
(m) & Q
\end{array}
$$

(page 375 of the book)

Context

The context of a line is the conjunction of all assumptions that have this line in their scope.

Convention: `True` denotes the empty context.

Validity

A formula at some line in a derivation is valid on this, and all lines below it, *until* the scope of one of the assumptions in its context ends.

Validity condition: rules may only be applied to valid formulas.

Introduction of negation

How do we establish a negation in a reasoning?

$\neg$-introduction:

$$
\begin{array}{ll}
 & \{\ \text{Assume:}\ \} \\
(k) & \boxed{P} \\
 & \vdots \\
(\ell-1) & \texttt{False} \\
 & \{\ \neg\text{-intro on } (k) \text{ and } (\ell-1):\ \} \\
(\ell) & \neg P
\end{array}
$$

($\uparrow$)

Elimination of negation

How can we use a negation in a reasoning?

$P \wedge \neg P \overset{val}{\models}$ `False`

¬-elimination:

	$\Vert\Vert$
(k)	$\neg P$
	$\Vert\Vert$
(ℓ)	P
	$\Vert\Vert$
	{ ¬-elim on (k) and (ℓ): }
(m)	`False`

$(\downarrow)$ $(k < m)$, $(\ell < m)$

Example

Show with a *derivation* (according to the methods of part II of the book) that the formula

$$(P \Rightarrow \neg Q) \Rightarrow (Q \Rightarrow \neg P)$$

is a tautology.

[Derivation on blackboard]

Introduction of `False`

How can we establish `False` *in a reasoning?*

$P \wedge \neg P \overset{val}{\models}$ `False`

NB: `False`-introduction is essentially the same ¬-elimination.

Subtle difference: `False`-introduction is intended for reasoning from bottom to top, whereas ¬-elimination is intended for reasoning from top to bottom.

In practice, we will only use ¬-elimination, and never use `False`-introduction!

`False`-introduction:

	$\vdots$
(k)	$\neg P$
	$\vdots$
(ℓ)	P
	{ `False`-intro on (k) and (ℓ): }
(m)	`False`

$(\uparrow)$

Elimination of `False`

How can we use `False` *in a reasoning?*

`False` $\overset{val}{\models} P$

`False`-elimination:

	$\Vert\Vert$
(k)	`False`
	$\Vert\Vert$
	{ `False`-elim on (k): }
(ℓ)	P

$(\downarrow)$

An example of a mathematical proof

Theorem
For all $x \in \mathbb{Z}$, if x^2 is odd, then so is x.

Proof:
Let $x \in \mathbb{Z}$, and suppose x^2 is odd;
we need to prove that x is odd too.

To this end, suppose that x is even;
we derive a contradiction.

If x is even, then $x = 2y$ for some $y \in \mathbb{Z}$.
Then $x^2 = (2y)^2 = 4y^2 = 2(2y^2)$ and $2y^2 \in \mathbb{Z}$.

So x^2 is even, and we have our contradiction.

Proof by contradiction

	Assume:
(k)	$\neg P$
	$\vdots$
$(\ell - 1)$	`False`
	{ RAA* on (k) and $(\ell - 1)$: }
(ℓ)	P

* RAA: Reductio Ad Absurdum (book does *not* use this name)

Introduction of double negation

How can we establish $\neg\neg$ in a reasoning?

$$P \stackrel{val}{==} \neg\neg P$$

$\neg\neg$-introduction:

	$\vdots$
$(\ell - 1)$	P
	{ $\neg\neg$-intro on $(\ell - 1)$: }
(ℓ)	$\neg\neg P$

$(\uparrow)$

Elimination of double negation

How can we use $\neg\neg$ in a reasoning?

$$\neg\neg P \stackrel{val}{==} P$$

$\neg\neg$-elimination:

	$\vert\vert\vert$
(k)	$\neg\neg P$
	$\vert\vert\vert$
	{ $\neg\neg$-elim on (k): }
(ℓ)	P

$(\downarrow)$

Proof by contradiction via ¬¬-elim

1. Recall ¬-introduction:
2. Instantiate with $\neg P$ for P:
3. Apply ¬¬-elim:

The steps from $(\ell - 1)$ to $(\ell + 1)$ may be summarized as

{ ¬-intro on (k) and $(\ell - 1)$ followed by ¬¬-elim: }

or as

{ RAA on (k) and $(\ell - 1)$: }

proof by contradiction:

{ Assume: }

(k) $\neg P$

⋮

$(\ell - 1)$ `False`

{ ¬-intro on (k) and $(\ell - 1)$: }

(ℓ) $\neg\neg P$

{ ¬¬-elim on (ℓ): }

$(\ell + 1)$ P

$(\uparrow)$

Example

Show with a *derivation* (according to the methods of part II of the book) that the formula

$$(\neg P \Rightarrow (P \wedge Q)) \Rightarrow P$$

is a tautology.

[Derivation on blackboard]

Introduction of disjunction (idea)

Truth table for $\vee$:

P	Q	$P \vee Q$
0	0	0
0	1	1
1	0	1
1	1	1

If we want to establish that a disjunction $P \vee Q$ is valid.

Then it is enough to establish that one of its disjuncts (P or Q) is valid.

We may even do so assuming that the other disjunct is not valid.

Introduction of disjunction (rule)

$\vee$-introduction:

{ Assume: }

(k) $\neg P$

⋮

$(\ell - 1)$ Q

{ $\vee$-intro on (k) and $(\ell - 1)$: }

(ℓ) $P \vee Q$

$(\uparrow)$

{ Assume: }

(k) $\neg Q$

⋮

$(\ell - 1)$ P

{ $\vee$-intro on (k) and $(\ell - 1)$: }

(ℓ) $P \vee Q$

$(\uparrow)$

Elimination of disjunction (idea)

Truth table for $\vee$:

P	Q	$P \vee Q$
0	0	0
0	1	1
1	0	1
1	1	1

Consider the situation that a disjunction $P \vee Q$ is valid.

Then we cannot conclude anything about the disjuncts P and Q.

But if, in addition, one of the disjuncts, say P, is not valid, then we can conclude that the other disjunct, say Q, must be valid.

Elimination of disjunction

How do we use a disjunction in a reasoning?

$\vee$-elimination:

$$\begin{array}{ll} & \vdots \\ (k) & P \vee Q \\ & \vdots \\ (\ell) & \neg P \\ & \vdots \\ & \{\ \vee\text{-elim on } (k) \text{ and } (\ell)\text{: }\} \\ (m) & Q \end{array}$$

$(\downarrow)$ $(k < \ell)$

$$\begin{array}{ll} & \vdots \\ (k) & P \vee Q \\ & \vdots \\ (\ell) & \neg Q \\ & \vdots \\ & \{\ \vee\text{-elim on } (k) \text{ and } (\ell)\text{: }\} \\ (m) & P \end{array}$$

$(\downarrow)$ $(k < \ell)$

Introduction of bi-implication

How do we establish a bi-implication in a reasoning?

$$(P \Rightarrow Q) \wedge (Q \Rightarrow P) \stackrel{val}{==} P \Leftrightarrow Q \ .$$

$\Leftrightarrow$-introduction:

$$\begin{array}{ll} & \vdots \\ (k) & P \Rightarrow Q \\ & \vdots \\ (\ell) & Q \Rightarrow P \\ & \{\ \Leftrightarrow\text{-intro on } (k) \text{ and } (\ell)\text{: }\} \\ (m) & P \Leftrightarrow Q \end{array}$$

$(\uparrow)$ $(k < \ell)$

Elimination of bi-implication

How can we use a bi-implication in a reasoning?

$$P \Leftrightarrow Q \stackrel{val}{==} \begin{cases} P \Rightarrow Q \\ Q \Rightarrow P \end{cases}$$

$\Leftrightarrow$-elimination:

$$\begin{array}{ll} & \vdots \\ (k) & P \Leftrightarrow Q \\ & \vdots \\ & \{\ \Leftrightarrow\text{-elim on } (k)\text{: }\} \\ (\ell) & P \Rightarrow Q \end{array}$$

$(\downarrow)$ $(k < \ell)$

$$\begin{array}{ll} & \vdots \\ (k) & P \Leftrightarrow Q \\ & \vdots \\ & \{\ \Leftrightarrow\text{-elim on } (k)\text{: }\} \\ (\ell) & Q \Rightarrow P \end{array}$$

$(\downarrow)$ $(k < \ell)$

Example

Show with a *derivation* (according to the methods of part II of the book) that the formula

$$(P \Leftrightarrow Q) \Rightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$$

is a tautology.

This is Exercise 14.9(a). See the section with *Course Material* on the website for an incrementally built solution with detailed explanations.

Constructing a derivation

forward reasoning: draw conclusions from hypotheses or facts; and

backward reasoning: simplify one of the remaining goals (on basis of main symbol)

Main strategy:

Stick to backward reasoning until no remaining goals can be simplified further; then consider forward reasoning.

Main strategy works most of the time, and should always be tried first.

Fall-back strategies:

1. invent and prove new goal to be able to use hypotheses; or
2. apply a proof by contradiction.

NB: fall-back strategies should be used with caution, and **only** if the main strategy does not work!

Proving a universal quantification

How can we prove that $\forall_x[x \in \mathbb{Z} \wedge x \geq 3 : x^2 - x - 2 \neq 0]$?

Proof:

Let $x \in \mathbb{Z}$ such that $x \geq 3$.

Then, for that particular x, it holds that

$$x^2 - x - 2 = (x+1)(x-2) \ .$$

So $x^2 - x - 2 \neq 0$. (Why?)

Conclusion: $\forall_x[x \in \mathbb{Z} \wedge x \geq 3 : x^2 - x - 2 \neq 0]$.

Introduction of universal quantification

How do we establish a universal quantification in a reasoning?

$\forall$-introduction:

{ Assume: }

(k) **var** $x; P(x)$

$\vdots$

$(\ell - 1)$ $Q(x)$

{ $\forall$-intro on (k) and $(\ell - 1)$: }

(ℓ) $\forall_x[P(x) : Q(x)]$

$(\uparrow)$

NB: variable x declared in line (k) should be new!

- to prove the universal quantification

  $\forall_x[P(x) : Q(x)]$

- consider an arbitrary element x (i.e., a variable) satisfying P
- and prove that this element x also satisfies Q.

Using a universal quantification

Suppose we 'know' $\forall_x[x \in \mathbb{R} \wedge x > 2 : 1 \leq \frac{x+1}{x-1} < 3]$.

Then, whenever we encounter $a \in \mathbb{R}$ with $a > 2$, we may conclude $1 \leq \frac{a+1}{a-1} < 3$.

For instance:

- if $a = 5$, then, since $5 \in \mathbb{R}$ and $5 > 2$, we may conclude that $1 \leq \frac{5+1}{5-1} < 3$.
- if $a = \pi - 1$, then, since $\pi - 1 \in \mathbb{R}$ and $\pi - 1 > 2$, we may conclude that $1 \leq \frac{\pi}{\pi-2} < 3$.
- ...

Elimination of universal quantification

How do we use a universal quantification in a reasoning?

$\forall$-elimination:

	$\vert\vert\vert$
(k)	$\forall_x[P(x) : Q(x)]$
	$\vert\vert\vert$
(ℓ)	$P(a)$
	$\vert\vert\vert$
	{ $\forall$-elim on (k) and (ℓ): }
(m)	$Q(a)$

$(\downarrow)$

NB1: a is an 'object' (variable, number, ...) that is 'known' in line (ℓ).

NB2: the as occurring in lines (ℓ) and (m) denote the same object!

Renaming of bound variables

Example:

$\forall_m[m \in \mathbb{N} : m + n > 6]$ (true if $n > 6$; false if $n \leq 6$)

$\stackrel{val}{==} \forall_k[k \in \mathbb{N} : k + n > 6]$ (true if $n > 6$; false if $n \leq 6$)

$\stackrel{val}{\neq} \forall_n[n \in \mathbb{N} : n + n > 6]$ (always false)

It is allowed to rename bound variables also in a derivation (provided that the binding structure does not change).

In fact, before starting to construct a derivation, it is advisable to apply a renaming to the goal such that bound variables bound by different quantifiers are named differently.

Example

Show with a *derivation* (according to the methods of part II of the book) that the formula

$$\forall_x[\neg P(x) : Q(x)] \Rightarrow \forall_x[\neg Q(x) : P(x) \vee R(x)]$$

is a tautology.

[Derivation on blackboard]

Proving an existential quantification

Suppose we want to establish $\exists_x[x \in \mathbb{Z} : x^3 - 3x = 2]$.

Then it suffices to find a *witness*:

- 2 is a witness, because $2 \in \mathbb{Z}$ and $2^3 - 3 \cdot 2 = 2$;
- -1 is *also* a witness, because $-1 \in \mathbb{Z}$ and $(-1)^3 - 3 \cdot (-1) = 2$.

Introduction of ∃ (witness)

How do we establish an existential quantification in a reasoning?

$\exists^*$-introduction:

	⋮
(k)	$P(a)$
	⋮
(ℓ)	$Q(a)$
	{ $\exists^*$-intro on (k) and (ℓ): }
(m)	$\exists_x[P(x) : Q(x)]$

$(\uparrow)$ $(k < m)$, $(\ell < m)$

NB1: Strategy for existential quantification as a goal: wait for suitable a.

NB2: Some luck is needed here: the $\exists^*$-intro rule does not always work!

Using an existential quantification

Suppose we 'know' $\exists_x[x \in \mathbb{R} : a - x < 0 < b - x]$.

Then we can declare an $x \in \mathbb{R}$ such that $a - x < 0 < b - x$, and use it in our proof.

For instance:

- $a - x < 0$, so $a < x$; and
- $0 < b - x$, so $x < b$.

So $a < x < b$, and hence $a < b$.

Elimination of ∃ (witness)

How do we use an existential quantification in a reasoning?

$\exists^*$-elimination:

	⫴
(k)	$\exists_x[P(x) : Q(x)]$
	⫴
	{ $\exists^*$-elim on (k): }
(ℓ)	Pick an x with $P(x)$ and $Q(x)$

$(\downarrow)$

NB1: x has to be new.

Example

Show with a *derivation* (according to the methods of part II of the book) that the formula

$$\exists_x \forall_y [P(y) \Rightarrow Q(y,x)] \Rightarrow \forall_x [P(x) : \exists_y [Q(x,y)]]$$

is a tautology.

[Derivation on blackboard]

Case distinction

How can we prove the inequality $|x-1| \geq \frac{1}{2}x - \frac{1}{2}$ for all $x \in \mathbb{Z}$?

Consider an arbitrary $x \in \mathbb{Z}$. Note that

$$|x-1| = \begin{cases} x-1 & \text{if } x \geq 1 \quad \text{(case 1)} \\ -x+1 & \text{if } x \leq 1 \quad \text{(case 2)} \end{cases}.$$

We therefore distinguish two cases:

1. If $x \geq 1$, then $\frac{1}{2}x \geq \frac{1}{2}$, so $|x-1| = x-1 \geq \frac{1}{2}x - \frac{1}{2}$.
2. If $x \leq 1$, then $-\frac{3}{2}x \geq -\frac{3}{2}$, so $|x-1| = -x+1 \geq \frac{1}{2}x - \frac{1}{2}$.

In both cases the inequality holds, and (at least) one of the two cases applies for every $x \in \mathbb{Z}$. So the inequality holds for every $x \in \mathbb{Z}$.

Case distinction (rule)

Proof by case distinction:

	$\vdots$
(k)	$P \vee Q$
	$\vdots$
(ℓ)	$P \Rightarrow R$
	$\vdots$
(m)	$Q \Rightarrow R$
	$\vdots$
	{ Case distinction on (k), (ℓ) and (m): }
(n)	R

Justification

Show with a *derivation* (according to the methods of part II of the book) that

$$(P \vee Q) \wedge (P \Rightarrow R) \wedge (Q \Rightarrow R) \Rightarrow R \ .$$

[See book Section 14.7]

Introduction of ∃ (general)

How do we establish an existential quantification in a reasoning?

- to prove $\exists_x[P(x) : Q(x)]$
- assume $\forall_x[P(x) : \neg Q(x)]$
- and derive a contradiction

∃-introduction:

{ Assume: }

(k) $\forall_x[P(x) : \neg Q(x)]$

⋮

$(\ell - 1)$ `False`

{ ∃-intro on (k) and $(\ell - 1)$: }

(ℓ) $\exists_x[P(x) : Q(x)]$

$(\uparrow)$

Elimination of ∃ (general)

How do we use an existential quantification in a reasoning?

- if we know (in a certain context!) that $\exists_x[P(x) : Q(x)]$
- and we also know (in that context) that $\forall_x[P(x) : \neg Q(x)]$
- then we have a contradiction

∃-elimination:

$|||$

(k) $\exists_x[P(x) : Q(x)]$

$|||$

(ℓ) $\forall_x[P(x) : \neg Q(x)]$

$|||$

{ ∃-elim on (k) and (ℓ): }

(m) `False`

$(\downarrow)$