Relations

Lectures 8–9 (Chapter 17)

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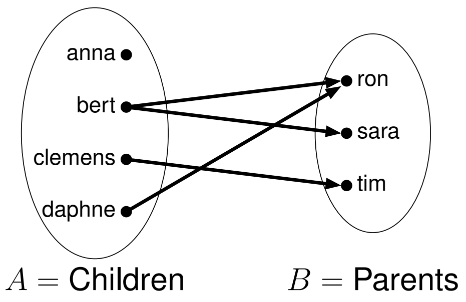
Relation between sets

A *relation* R *between sets* A *and* B *is a predicate on* $A \times B$.
$R(x, y)$ *means '*$x \in A$ *and* $y \in B$ *are related.'*

Example (Arrow Graph)



The relation 'is child of' is a predicate on

Children $\times$ Parents .

In this example we have:

$R(b, r)$, $R(b, s)$, $R(c, t)$, and $R(d, r)$

We sometimes write bRr instead of $R(b, r)$.

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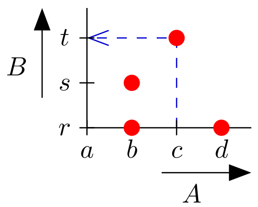
Relation between sets

A *relation* R *between sets* A *and* B *is a subset of* $A \times B$.
That is: $R \subseteq A \times B$.

Example (Cartesian Graph)



$$R = \{(b, r), (b, s), (c, t), (d, r)\} \ .$$

We sometimes write $(b, r) \in R$ instead of $R(b, r)$.

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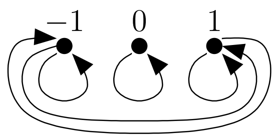
Relation on a set

A *relation* R *on* A *is a relation between* A *and* A.

Example (Directed Graph)

R on $\{-1, 0, 1\}$ with $R(x, y)$ if $|x| = |y|$:



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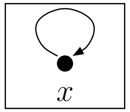
Reflexive relations

A relation R on A is reflexive if

$$\forall_x[x \in A : xRx] \ .$$



for all $x \in A$.

Reflexive:

- $\stackrel{val}{==}$, $\stackrel{val}{\models}$ on abstr. props
- $=, \geq$ on $\mathbb{R}$
- $\ldots$

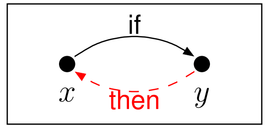
Not reflexive:

- $>$ on $\mathbb{R}$
- R on $\mathbb{N}$ with xRy if $x = 3y$
  (Counterexample: $\neg(1R1)$.)
- $\ldots$

Symmetric relations

A relation R on A is symmetric if

$$\forall_{x,y}[x, y \in A : xRy \Rightarrow yRx] \ .$$



for all $x, y \in A$.

Symmetric:

- $\stackrel{val}{==}$ on abstr. props
- $=$ on $\mathbb{R}$
- '*is married*' on people
- $\ldots$

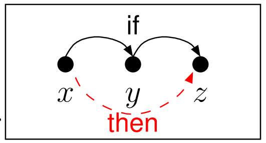
Not symmetric:

- $\stackrel{val}{\models}$ on abstr. props
- $>, \geq$ on $\mathbb{R}$
- '*is child of*' on people
- R on $\mathbb{N}$ with xRy if $x = 3y$
  (Counterexample: $6R2$, but $\neg(2R6)$)
- $\ldots$

Transitive relations

A relation R on A is transitive if

$$\forall_{x,y,z}[x, y, z \in A : (xRy \wedge yRz) \Rightarrow xRz] \ .$$



for all $x, y, z \in A$.

Transitive:

- $\stackrel{val}{==}$, $\stackrel{val}{\models}$ on abstr. props
- $=, >, \geq$ on $\mathbb{R}$
- $\ldots$

Not transitive:

- 'is sister of' on people
- R on $\mathbb{N}$ with xRy if $x = 3y$
  (Counterexample: $18R6$ and $6R2$, but $\neg(18R2)$)
- $\ldots$

Equivalence relation

R on A is an equivalence relation if and only if

1. R is reflexive, **and**
2. R is symmetric, **and**
3. R is transitive.

Examples

- $\stackrel{val}{==}$ on abstr. props
- $=$ on $\mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{C}, \ldots$
- $=$ on sets
- the relation $\equiv_6$ on $\mathbb{Z}$ defined by

  $x \equiv_6 y$ if $x - y$ is a multiple of 6 → $\exists_k[k \in \mathbb{Z} : x - y = k \cdot 6]$

  (e.g., $10 \equiv_6 4$, $61 \equiv_6 1$, $-2 \equiv_6 10$, $9 \equiv_6 9$)

$\equiv_6$ on $\mathbb{Z}$

Define the binary relation $\equiv_6$ on $\mathbb{Z}$ by

$x \equiv_6 y$ if $\exists_k[k \in \mathbb{Z} : x - y = 6 \cdot k]$.

Fact

$\equiv_6$ is an equivalence relation on $\mathbb{Z}$.

Proof:

We need to prove that

1. $\equiv_6$ is reflexive;
   [Proof: see book.]
2. $\equiv_6$ is symmetric;
   [Proof: see book.]
3. $\equiv_6$ is transitive.
   [Proof on next slide; see also book]

$\equiv_6$ is transitive:

var $x, y, z;\ x, y, z \in \mathbb{Z}$

$x \equiv_6 y \wedge y \equiv_6 z$

$\exists_k[k \in \mathbb{Z} : x - y = 6 \cdot k]$	($\wedge$-elim + definition $\equiv_6$)
Pick a k with $k \in \mathbb{Z}$ and $x - y = 6 \cdot k$	($\exists^*$-elim)
$\exists_\ell[\ell \in \mathbb{Z} : y - z = 6 \cdot \ell]$	($\wedge$-elim + definition $\equiv_6$)
Pick an ℓ with $\ell \in \mathbb{Z}$ and $y - z = 6 \cdot \ell$	($\exists^*$-elim)
$x - z = (x - y) + (y - z) = 6 \cdot k + 6 \cdot \ell = 6 \cdot (k + \ell)$	(Mathematics)
$x - z = 6 \cdot (k + \ell)$	(Mathematics)
$k + \ell \in \mathbb{Z}$	(since $k \in \mathbb{Z}$ and $\ell \in \mathbb{Z}$)
$\exists_m[m \in \mathbb{Z} : x - z = 6 \cdot m]$	($\exists^*$-intro)
$x \equiv_6 z$	(definition $\equiv_6$)

$\forall_{x,y,z}[x, y, z \in \mathbb{Z} : x \equiv_6 y \wedge y \equiv_6 z \Rightarrow x \equiv_6 z]$

Note: It is important to choose k and ℓ distinct in this proof.

$\equiv_6$ is transitive:

Proof:

To prove that $x \equiv_6 y$ and $y \equiv_6 z$ implies $x \equiv_6 z$ for all $x, y, z \in \mathbb{Z}$, let $x, y, z \in \mathbb{Z}$ and suppose that $x \equiv_6 y$ and $y \equiv_6 z$; we need to establish that $x \equiv_6 z$.

From $x \equiv_6 y$ and $y \equiv_6 z$ it follows, by the definition of $\equiv_6$, that there exists k and ℓ with $x - y = 6 \cdot k$ and $y - z = 6 \cdot \ell$.

Hence,

$$x - z = (x - y) + (y - z) = 6 \cdot k + 6 \cdot \ell = 6 \cdot (k + \ell) \ .$$

Since $k, \ell \in \mathbb{Z}$, we also have that $(k + \ell) \in \mathbb{Z}$.

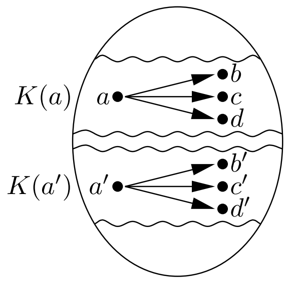
So we have that $x - z$ is a multiple of 6, and hence, according to the definition of $\equiv_6$, it follows that $x \equiv_6 z$.

Equivalence classes

Let A be a set, let R be an equivalence relation on A.

$$K(a) \stackrel{\text{def}}{=} \{x \in A \mid aRx\}$$

The equivalence class of $a \in A$ is the set of all end-points of arrows from a.



Since R is an equivalence relation, everything within a class is related.

For instance:

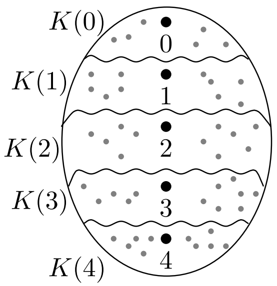
$$\left.\begin{array}{l} aRd \stackrel{\text{symm.}}{\rightsquigarrow} dRa \\ aRc \end{array}\right\} \stackrel{\text{trans.}}{\rightsquigarrow} dRc$$

Note:
K(a)=K(b)=K(c)=K(d)

Equivalence classes

Example

Consider the equivalence relation $\equiv_5$ on $\mathbb{Z}$.



Then, e.g.,

$$K(3) = \{\ldots, -7, -2, 3, 8, 13, \ldots\}\ .$$

Note: $K(4) = K(9) = K(14) = \cdots$

Equivalence classes

Example

Let R be an equivalence relation on a set A, and let $a, a' \in A$.

Prove that

$$\neg(aRa') \Rightarrow K(a) \cap K(a') = \emptyset$$

[Proof on next slide]

Example: $\neg(aRa') \Rightarrow K(a) \cap K(a') = \emptyset$

(1)	$\neg(a\ R\ a')$	
(2)	**var** x; $x \in K(a) \cap K(a')$	
(3)	$x \in K(a) \wedge x \in K(a')$	
(4)	$a\ R\ x$	
(5)	$a'\ R\ x$	
(6)	$x\ R\ a'$	($\forall$-elim on 'R is symmetric' followed by $\Rightarrow$-elim)
(7)	$a\ R\ a'$	($\forall$-elim on 'R is transitive' followed by $\wedge$-intro and $\Rightarrow$-elim)
(8)	`False`	
(9)	$K(a) \cap K(a') = \emptyset$	
(10)	$\neg(aRa') \Rightarrow K(a) \cap K(a') = \emptyset$	