Sets

Lectures 7 and 8 (Chapter 16)

Sets

We will not give a precise definition of what is *a set, but we will say precisely what you can* do *with it.*

(Think of a set as a collection of things of which order and multiplicity do not matter.)

Examples:

- $\{0, 1, 2, 3\} = \{3, 1, 0, 2\} = \{3, 1, 1, 1, 0, 0, 2\}$.
- $\{6\}$ is the singleton set (NB: $\{6\} \neq 6$).
- $\emptyset$ denotes the empty set (i.e., the set without elements).
- $\mathbb{N}$, $\mathbb{Z}$: standard set-theoretic notations for sets of numbers.

Specifying sets, membership

We write $t \in X$ for "t is an element of the set X".

Specifying a set using a predicate

$\{x \in D \mid P(x)\}$: the set of all $x \in D$ such that $P(x)$.

Examples

- $\{n \in \mathbb{N} \mid n > 10\}$ is the set of all natural numbers greater than 10;
- $\{x \in \mathbb{Z} \mid x^2 + x \leq 0\} = \{-1, 0\}$.

Membership

The element-of predicate $\in$ is a binary predicate.

Property of $\in$:
$t \in \{x \in D \mid P(x)\} \stackrel{val}{==} t \in D \wedge P(t)$

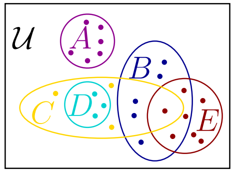
Subset, universe

$A \subseteq B \stackrel{\text{def}}{=} \forall_x[x \in A : x \in B]$

A is a subset of B (notation: $A \subseteq B$) if every element of A is also an element of B

Property of $\subseteq$:
$(A \subseteq B) \wedge t \in A \stackrel{val}{\models} t \in B$

The universe, denoted by $\mathcal{U}$, is a set of which all sets *in a particular context* are subsets.



Property of $\mathcal{U}$:
$t \in \mathcal{U} \stackrel{val}{==} \texttt{True}$

Equality of sets

Two sets are equal if they have exactly the same elements.

Example

$\{x \in \mathbb{Z} \mid x > 0\} = \{n \in \mathbb{N} \mid n \neq 0\}$

$A = B \stackrel{\text{def}}{=} A \subseteq B \wedge B \subseteq A$

Property of $=$:

$A = B \stackrel{val}{=\!=} \forall_x [x \in A \Leftrightarrow x \in B]$

$A = B \wedge t \in A \stackrel{val}{\models} t \in B$

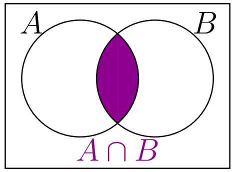
$A = B \wedge t \in B \stackrel{val}{\models} t \in A$

Leibniz for $=$:

$$\frac{A = B}{\cdots A \cdots = \cdots B \cdots}$$

Intersection

The intersection of A and B is the set of everything that is both in A and in B



$A \cap B \stackrel{\text{def}}{=} \{x \in \mathcal{U} \mid x \in A \wedge x \in B\}$

Example

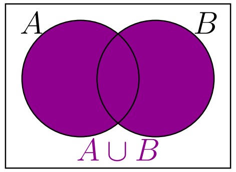
$\{n \in \mathbb{N} \mid n > 5\} \cap \{n \in \mathbb{N} \mid n < 10\} = \{6, 7, 8, 9\}$

Property of $\cap$:

$t \in A \cap B \stackrel{val}{=\!=} t \in A \wedge t \in B$

Union

The union of A and B is the set of everything that is in A or in B



$A \cup B \stackrel{\text{def}}{=} \{x \in \mathcal{U} \mid x \in A \vee x \in B\}$

Example

$\{n \in \mathbb{N} \mid n > 5\} \cup \{n \in \mathbb{N} \mid n < 10\} = \mathbb{N}$

Property of $\cup$:

$t \in A \cup B \stackrel{val}{=\!=} t \in A \vee t \in B$

Commutativity and associativity of $\cap$ and $\cup$

- $\cap$ and $\cup$ are commutative:

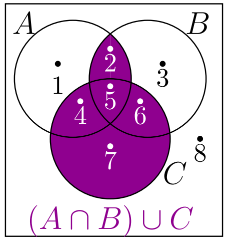
  $A \cap B = B \cap A$, and
  $A \cup B = B \cup A$;

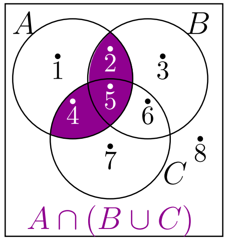
- $\cap$ and $\cup$ are associative:

  $(A \cap B) \cap C = A \cap (B \cap C)$, and
  $(A \cup B) \cup C = A \cup (B \cup C)$.

Example

$(A \cap B) \cup C = A \cap (B \cup C)$ does not hold for all sets A, B and C!





Counterexample:

Let $A = \{1, 2, 4, 5\}$, $B = \{2, 3, 5, 6\}$ and $C = \{4, 5, 6, 7\}$.

Then $A \cap B = \{2, 5\}$, $(A \cap B) \cup C = \{2, 4, 5, 6, 7\}$,
$B \cup C = \{2, 3, 4, 5, 6, 7\}$, and $A \cap (B \cup C) = \{2, 4, 5\}$.

So $(A \cap B) \cup C = \{2, 4, 5, 6, 7\} \neq \{2, 4, 5\} = A \cap (B \cup C)$.

Reasoning with the subset predicate

$\subseteq$-introduction:

	{ Assume: }
(k)	**var** x; $x \in A$
	$\vdots$
$(\ell - 2)$	$x \in B$
	{ $\forall$-intro on (k) and $(\ell - 2)$: }
$(\ell - 1)$	$\forall_x[x \in A : x \in B]$
	{ Definition of $\subseteq$ on $(\ell - 1)$: }
(ℓ)	$A \subseteq B$

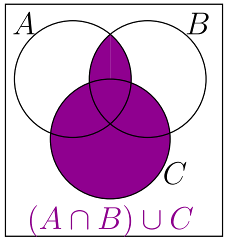
$(\uparrow)$

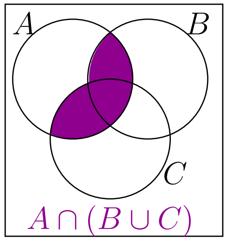
$\subseteq$-elimination:

	$\vdots$
(k)	$A \subseteq B$
	$\vdots$
(ℓ)	$t \in A$
	$\vdots$
	{ Property of $\subseteq$ on (k) and (ℓ): }
(m)	$t \in B$

$(\downarrow)$

Example





Fact

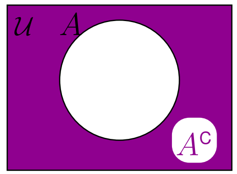
$A \cap (B \cup C) \subseteq (A \cap B) \cup C$ does hold.

[Proof on blackboard]

Complement

The complement A^{C} of A is the set of everything not in A

$$A^{\mathsf{C}} \stackrel{\text{def}}{=} \{x \in \mathcal{U} \mid \neg(x \in A)\}$$



Example

(Suppose that $\mathcal{U} = \mathbb{Z}$)
$\{x \in \mathbb{Z} \mid x \geq 0\}^{\mathsf{C}} = \{x \in \mathbb{Z} \mid x < 0\}$

Property of $^{\mathsf{C}}$:

$$t \in A^{\mathsf{C}} \stackrel{val}{==} \neg(t \in A)$$

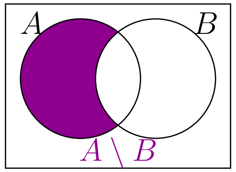
NB: For computing $^{\mathsf{C}}$ it is important to know what is the universe:

- If $\mathcal{U} = \mathbb{N}$, then $\{0, 1\}^{\mathsf{C}} = \{n \in \mathbb{N} \mid n \geq 2\}$
- If $\mathcal{U} = \mathbb{Z}$, then $\{0, 1\}^{\mathsf{C}} = \{x \in \mathbb{Z} \mid x < 0 \vee x \geq 2\}$.

Difference

The difference of A and B is the set of everything that is in A, but not also in B

$$A \setminus B \stackrel{\text{def}}{=} \{x \in \mathcal{U} \mid x \in A \wedge \neg(x \in B)\}$$



Example

$\{n \in \mathbb{N} \mid n > 5\} \setminus \{n \in \mathbb{N} \mid n < 10\} = \{n \in \mathbb{N} \mid n \geq 10\}$

Property of $\setminus$:

$$t \in A \setminus B \stackrel{val}{==} t \in A \wedge \neg(t \in B)$$

Equality of sets (reasoning)

=-introduction:

$$\begin{array}{ll} & \vdots \\ (k) & A \subseteq B \\ & \vdots \\ (\ell-2) & B \subseteq A \\ & \{\ \wedge\text{-intro on } (k) \text{ and } (\ell-2)\text{: } \} \\ (\ell-1) & A \subseteq B \wedge B \subseteq A \\ & \{\text{ Definition of } = \text{ on } (\ell-1)\text{: } \} \\ (\ell) & A = B \end{array}$$

$(\uparrow)$

=-elimination:

$$\begin{array}{ll} & ||| \\ (k) & A = B \\ & ||| \\ & \{\text{ Definition of } = \text{ on } (k)\text{: } \} \\ (\ell) & A \subseteq B \wedge B \subseteq A \end{array}$$

$(\downarrow)$

Examples

Assume $\mathcal{U} = \mathbb{Z}$.

- Prove that $A \setminus B^{\mathsf{c}} = A \cap B$ for all sets A and B.

  [Proof on blackboard]

- Determine for each of the following formulas whether it holds for all sets A and B. If so, then give a proof; if not, then give a counterexample:
  - $A \setminus B = A \ \Rightarrow\ A = B^{\mathsf{c}}$;
  - $A = B^{\mathsf{c}} \ \Rightarrow\ A \setminus B = A$.

Notation

$0 \notin \{\{0\}\}$	$0 \not\subseteq \{\{0\}\}$
$\{0\} \in \{\{0\}\}$	$\{0\} \not\subseteq \{\{0\}\}$
$\emptyset \notin \{\{0\}\}$	$\emptyset \subseteq \{\{0\}\}$
$\{\{0\}\} \notin \{\{0\}\}$	$\{\{0\}\} \subseteq \{\{0\}\}$

Empty set

The *empty set* $\emptyset$ *is the unique set without elements.*

$\emptyset = \{x \in \mathcal{U} \mid \texttt{False}\}$

Property of $\emptyset$:
$t \in \emptyset \stackrel{val}{==} \texttt{False}$

$\emptyset \subseteq A \stackrel{val}{==} \texttt{True}$

$\emptyset \subseteq A$
$\stackrel{val}{==}$ { Definition $\subseteq$ }
$\forall_x[x \in \emptyset : x \in A]$
$\stackrel{val}{==}$ { Property of $\emptyset$ }
$\forall_x[\texttt{False} : x \in A]$
$\stackrel{val}{==}$ { Empty domain }
$\texttt{True}$

$A \subseteq \emptyset \stackrel{val}{==} \forall_x[x \in A : \texttt{False}]$

$A \subseteq \emptyset$
$\stackrel{val}{==}$ { Definition of $\subseteq$ }
$\forall_x[x \in A : x \in \emptyset]$
$\stackrel{val}{==}$ { Property of $\emptyset$ }
$\forall_x[x \in A : \texttt{False}]$

Alternative property of empty set

From previous slide:

(*) $\emptyset \subseteq A \stackrel{val}{==} \texttt{True}$ (**) $A \subseteq \emptyset \stackrel{val}{==} \forall_x[x \in A : \texttt{False}]$

So:

$A = \emptyset$
$\stackrel{val}{==}$ { Definition of $=$ }
$A \subseteq \emptyset \wedge \emptyset \subseteq A$
$\stackrel{val}{==}$ { (*) + $\texttt{True}$/$\texttt{False}$-elimination }
$A \subseteq \emptyset$
$\stackrel{val}{==}$ { (**) }
$\forall_x[x \in A : \texttt{False}]$

Property of $\emptyset$:
$A = \emptyset \stackrel{val}{==} \forall_x[x \in A : \texttt{False}]$

Example

Prove that $A \cup B = A \Rightarrow B \setminus A = \emptyset$ for all sets A and B.

[Proof on blackboard. (Also available as detailed example of a derivation-style proof of a set-theoretic property from Course Material section of the website)]

Powerset

The *powerset* $\mathcal{P}(A)$ *of* A *is the set of all subsets of* A*.*

Examples

- $\mathcal{P}(\{4,6\}) = \{\emptyset, \{4\}, \{6\}, \{4,6\}\}$
- $\mathcal{P}(\{1,2,3\}) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$
- $\emptyset \in \mathcal{P}(\mathbb{R})$, $[0,5] \in \mathcal{P}(\mathbb{R})$
- $1 \notin \mathcal{P}(\mathbb{R})$, $\{1\} \in \mathcal{P}(\mathbb{R})$, $\{2,5,7\} \in \mathcal{P}(\mathbb{R})$
- $\mathbb{N} \in \mathcal{P}(\mathbb{R})$, $\mathbb{R} \in \mathcal{P}(\mathbb{R})$
- $\mathcal{P}(\{\emptyset, \{\emptyset\}\}) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$

If $\#A = n$, then $\#\mathcal{P}(A) = 2^n$.

Powerset in proofs

Property of $\mathcal{P}$:
$C \in \mathcal{P}(A) \overset{val}{=\!=} C \subseteq A$

Examples

- Prove that $\mathcal{P}(A) \subseteq \mathcal{P}(A \cup B)$ for all sets A and B. [Exercise]
- Does $\mathcal{P}(A) \setminus \mathcal{P}(B) \subseteq \mathcal{P}(A \setminus B)$ hold for all sets A and B?
  [Answer on next slide]

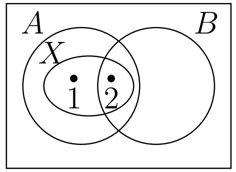
$\mathcal{P}(A) \setminus \mathcal{P}(B) \not\subseteq \mathcal{P}(A \setminus B)$

$$\begin{array}{ll} & X \in \mathcal{P}(A) \setminus \mathcal{P}(B) \\ \overset{val}{=\!=} & \{ \text{ Property of } \setminus \} \\ & X \in \mathcal{P}(A) \wedge \neg(X \in \mathcal{P}(B)) \\ \overset{val}{=\!=} & \{ \text{ Property of } \mathcal{P} \ (2\times) \} \\ & X \subseteq A \wedge \neg(X \subseteq B) \end{array}$$

$$\begin{array}{ll} & X \in \mathcal{P}(A \setminus B) \\ \overset{val}{=\!=} & \{ \text{ Property of } \mathcal{P} \} \\ & X \subseteq A \setminus B \end{array}$$

Counterexample:

Let $A = \{1, 2\}$, $B = \{2\}$ and $X = \{1, 2\}$.
Then $X \subseteq A$, so $X \in \mathcal{P}(A)$.
And $\neg(X \subseteq B)$, so $\neg(X \in \mathcal{P}(B))$.
Hence, $X \in \mathcal{P}(A) \setminus \mathcal{P}(B)$.
On the other hand, $A \setminus B = \{1\}$, so $\neg(X \subseteq A \setminus B)$.
Hence, $X \notin \mathcal{P}(A \setminus B)$.



Cartesian product

The Cartesian product $A \times B$ is the set of pairs (a, b) with $a \in A$ and $b \in B$.

Examples

- $\{0, 1\} \times \{3, 5, 7\} = \{(0, 3), (0, 5), (0, 7), (1, 3), (1, 5), (1, 7)\}$
- $\mathbb{N} \times \mathbb{Z} = \{(n, x) \mid n \in \mathbb{N} \wedge x \in \mathbb{Z}\}$
- $(3, -2) \in \mathbb{N} \times \mathbb{Z}$
- $(-2, 3) \notin \mathbb{N} \times \mathbb{Z}$

NB: $A^2 = A \times A$, $A^3 = A \times A \times A$, etc.

Cartesian product in proofs

Property of $\times$:
$(a, b) \in A \times B \overset{val}{=\!=} a \in A \wedge b \in B$

Example

Prove that $A \subseteq B \Rightarrow A^2 \subseteq A \times B$ for all sets A and B.

[See Section 16.9 of the book for the construction of a very similar proof.]

Mind the Brackets

In set theory, brackets have a meaning, and different brackets have different meanings!

0	:	a number
$\{0\}$	:	a set (containing a number)
$\{\{0\}\}$	:	a set (containing a set containing a number)
$\{0,1\}$	:	a set (containing two numbers, order does not matter)
$(0,1)$	:	a pair of numbers (order does matter!)
$\{(0,1)\}$	:	a set (containing a pair of numbers)
$\emptyset$	:	a set (containing nothing)
$\{\emptyset\}$	:	a set (containing the empty set)

Proofs in Set Theory

A proof of a set-theoretic property is a *convincing* argument based on given definitions and properties of sets (see table on p. 381*)

The logical reasoning structure of the proof must be clear and valid, but you may omit references to the names of the *logical reasoning steps* ($\wedge$-intro, $\Rightarrow$-elim, $\forall$-elim, etc.).

You may also mix styles (derivations, calculations, natural language).

You should, however, explicitly refer to the *set-theoretic properties* (Property of $\cap$, Definition of $\subseteq$, etc.) in your proof.

* Leibniz for equality of sets may be used as well, although it is not in the table.

Counterexamples in Set Theory

To refute the validity of a set-theoretic property, it is *not* enough to provide a diagram, or an informal reasoning of another kind.

A counterexample consists of

1. clear and concrete declarations of the sets involved (e.g., *let* $A = \ldots$, *let* $B = \ldots$);
2. evaluation of the expressions involved (e.g., *then* $A \cup B = \ldots$);
3. a convincing argument why the property is refuted (e.g., *since the left-hand side of the implication is true, but the right-hand side is not, it follows that the implication is false*).